Abstract Algebra

a journey through form

Dilip

Do Forward!

Abstract Algebra

A Journey through form

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De work is meant to be dreamt through, rather than read through, ponder each idea – not so much in the sense of a hurried math student. In the sense of a dreamer, dreaming. The stars in the sky are objects of the dream as well. Perhaps a greater insight awaits the dreamer

No effort has been made to make the text cover every major theorem and prove major result. When an idea is encountered – the reader us urged either to pause and dream in the idea, or to form a vague image and read on. Spend time to research the ideas and theorems presented, not in the sense of seeking a perfect mathematical understanding in the usual university sense, a few pattern repetition skills gained and few ideas of depth if any encountered.

Instead as every paragraph is an opportunity to stay

enchanted, stay wid de page.

Read a paragraph a day. Dream on. Discover what awareness dynamics are set into animation in these ideas.

Dilip

16th December, 2025.

Volume 1: Void, Symmetry, and the Primacy of Action

Ἐν ἀρχῇ ἦν ὁ λόγος

In the beginning was not the thing,

but the act of transformation—

the verb before the noun,

the doing before the being.

On Void:

Before structure, 空 (kū)—emptiness.
Not absence, but mathematical and super-mathematical pleasurous potential.
Not nothing, but the space where everything waits.
The empty set contains no elements,
yet holds within it every future construction.
Zero is not void of meaning—
it is origin, the axis from which the number, as symmetry, throwing again into the void, origins.

The Japanese speak of *間* (ma)—the space between. The silence between notes that makes music possible. The pause between breaths where awareness lives. In mathematics, we encounter this same principle: the unmarked state before symmetry breaks it, the neutral element before operation transforms it, the identity before action reveals structure.

When Thales contemplated the primordial waters, he sought not a substance but a principle—that from which forms emerge and to which they return. The group is such a principle: not a collection of objects, but the void in an ontological own awareness driven form-drive, the potential for action, the space where operations can occur.

On Symmetry:

Symmetry is not decoration. It is not the beauty of mirrors reflecting left to right. Symmetry is *that which remains unchanged when everything else transforms*.

The snowflake rotates—its form persists.
The equation transforms—its truth endures.
The object moves—its essence stays.
What is preserved reveals what *is*.

Here lies a paradox the Greeks understood: we know a thing not by what it is in itself, but by what survives when we act upon it. The sphere is known by its invariance under rotation. The circle by its perfect equidistance. Identity emerges through transformation—the self appears only in the mirror of change.

Symmetry is the answer to Parmenides' puzzle: how can there be both Being and Becoming? How can things change yet remain themselves? The group captures this precisely: elements that transform while structure persists, operations that alter while patterns endure. In mathematical symmetry, Heraclitus and Parmenides are reconciled.

On Action:

A group is not a noun—it is a verb disguised in nominal form.

We are taught to think in objects: sets of things, collections of elements, gatherings of entities. But this is backward. The group is first and foremost an *action*: a way of combining, a method of transforming, a pattern of relating. The elements exist only as placeholders for this action. The operation is primary; the operated-upon is secondary.

Not "things that can be combined,"
but "combination that requires things."
Not "elements with structure,"
but "structure that manifests as elements."
The dance precedes the dancers.
The transformation precedes the transformed.

This inversion of thought—from object to action, from being to doing—is perhaps the most profound shift mathematics offers. When we study group actions, we are not watching groups "act upon" passive sets. We are seeing action itself take form, transformation made tangible, the verb incarnate.

The Convergence:

From void springs the possibility of distinction. From distinction, symmetry emerges as that which preserves through change. From symmetry, action reveals itself as the fundamental reality. This is not a linear sequence but a three-fold unity—void, symmetry, action intertwined like the three strands of a rope, each supporting the others.

The empty set is the identity element writ large: that which, combined with anything, leaves it unchanged. Action is symmetry in

motion: the transformation that reveals what remains. Symmetry is the void's first differentiation: the breaking that preserves.

In this volume, we learn a new language—or rather, we learn that mathematics *is* language, but language stripped to its logical skeleton. We encounter the group: not as abstraction remote from experience, but as the structure underlying all pattern, all regularity, all form. Through subgroups we see how complexity nests within simplicity. Through cosets we learn how space can be partitioned without loss. Through homomorphisms we discover that structure itself can be mapped, preserved, transmitted across domains.

> **Greek Wisdom — Ἑλληνική Σοφία**
>
> *"γνῶθι σεαυτόν" (gnōthi seauton) — Know thyself*
>
> The Delphic maxim takes new meaning here: to know a mathematical object is to know its symmetries, its automorphisms, its group of self-transformations. What remains unchanged when a thing acts upon itself? This is its essence. Identity is not static—it is the invariant core around which transformation pivots. To know thyself is to know which transformations leave you unchanged, which operations preserve your structure. The study of groups is the study of identity through the lens of action.

> **Zen Reflection — 禅の省察**

> *Form is emptiness; emptiness is form. Form does not differ from emptiness; emptiness does not differ from form.*
>
> The Heart Sutra's paradox is the mathematician's daily experience. The group is simultaneously empty—a mere formal structure, symbols on a page—and full of concrete meaning, manifesting in rotations, permutations, transformations. The Zen adept speaks of 初心 (shoshin), beginner's mind: approaching each moment without preconception. So must we approach these foundations—not as preliminaries to rush through, but as koans to contemplate. What is the sound of one hand clapping? What is the identity element of your own consciousness?

Enter now the void that is not empty, the symmetry that is not stillness, the action that is not motion. Here, at the origin of structure itself, where zero fountains into infinite possibility, the foundations await not as solid ground but as the space in which all construction becomes possible.

❦

Chapter 1: Mathematical Foundations

The edifice of abstract algebra rests upon a foundation of fundamental mathematical concepts and techniques. Like the ancient Greek mathematicians who sought to understand the eternal forms beneath the changing phenomena, we begin our journey by examining the basic structures that underlie all of algebra: sets, functions, relations, and the art of mathematical proof itself.

This chapter establishes the language and tools we shall employ throughout our exploration of algebraic structures. We approach these foundations with both rigor and insight, for a deep understanding of these preliminaries illuminates the path ahead and reveals the profound unity underlying seemingly disparate algebraic concepts.

1.1 Set Theory

Set theory provides the fundamental language of modern mathematics. While we employ naive set theory for our purposes, we acknowledge the axiomatic foundations established by Zermelo-Fraenkel with the Axiom of Choice (ZFC).

> **Definition**
>
> A **set** is a well-defined collection of distinct objects, called **elements** or **members**. If x is an element of set A, we write $x \in A$. If x is not an element of A, we write $x \notin A$.

Sets may be specified by listing their elements, as in $A = \{1, 2, 3\}$, or by stating a property that characterizes membership, as in $B = \{x \in \mathbb{Z} : x^2 < 10\}$. The fundamental principle is that a set is completely determined by its elements—two sets are equal if and only if they contain precisely the same elements.

> **Definition**
>
> Let A and B be sets.
>
> - The **union** of A and B is $A \cup B = \{x : x \in A \text{ or } x \in B\}$.
> - The **intersection** of A and B is $A \cap B = \{x : x \in A \text{ and } x \in B\}$.
> - The **difference** of A and B is $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$.
> - The **complement** of A (relative to a universal set U) is $A^c = U \setminus A$.
> - A is a **subset** of B, written $A \subseteq B$, if every element of A is also an element of B.

Theorem

(De Morgan's Laws) For any sets A and B,

$$(A \cup B)^c = A^c \cap B^c \tag{1}$$
$$(A \cap B)^c = A^c \cup B^c \tag{2}$$

Proof. We prove the first equality; the second follows similarly. Let $x \in (A \cup B)^c$. Then $x \notin A \cup B$, which means $x \notin A$ and $x \notin B$. Thus $x \in A^c$ and $x \in B^c$, so $x \in A^c \cap B^c$. Conversely, if $x \in A^c \cap B^c$, then $x \in A^c$ and $x \in B^c$, meaning $x \notin A$ and $x \notin B$. Therefore $x \notin A \cup B$, so $x \in (A \cup B)^c$. ∎

Definition

Let A be a set.

- The **power set** of A, denoted $\mathcal{P}(A)$ or 2^A, is the set of all subsets of A:

$$\mathcal{P}(A) = \{B : B \subseteq A\}$$

- The **Cartesian product** of sets A and B is

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

Example

If $A = \{1, 2\}$, then $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, which has 4 elements. Note that $|\mathcal{P}(A)| = 2^{|A|} = 2^2 = 4$. Similarly, $A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$.

Cardinality

The concept of cardinality extends our intuitive notion of "size" to infinite sets, revealing profound insights about the nature of infinity itself.

Definition

Two sets A and B have the same **cardinality**, written $|A| = |B|$ or $A \sim B$, if there exists a bijection $f : A \to B$. A set is:

- **Finite** if it has the same cardinality as $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$.
- **Countably infinite** if it has the same cardinality as $\mathbb{N}$.
- **Countable** if it is finite or countably infinite.
- **Uncountable** if it is infinite but not countably infinite.

Example

The set of even integers $2\mathbb{Z}$ is countably infinite, as the function $f : \mathbb{N} \to 2\mathbb{Z}$ defined by $f(n) = 2n$ is a bijection. Similarly, $\mathbb{Z}$ is countable via the bijection:

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ -(n-1)/2 & \text{if } n \text{ is odd} \end{cases}$$

This gives the enumeration: $0, -1, 1, -2, 2, -3, 3, \ldots$

Theorem

(Cantor's Theorem) For any set A, there is no surjection from A onto $\mathcal{P}(A)$. In particular, $|A| < |\mathcal{P}(A)|$.

> ***Proof.*** Suppose, for contradiction, that $f : A \to \mathcal{P}(A)$ is surjective. Define $B = \{x \in A : x \notin f(x)\}$. Since f is surjective, there exists $b \in A$ such that $f(b) = B$. Now consider whether $b \in B$. If $b \in B$, then by definition of B, we have $b \notin f(b) = B$, a contradiction. If $b \notin B$, then $b \notin f(b)$, so by definition of B, we have $b \in B$, again a contradiction. This contradiction shows no such surjection exists. ∎

Cantor's theorem reveals the existence of infinitely many different "sizes" of infinity, forming a hierarchy: $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \cdots$

1.2 Functions and Relations

Functions provide the means by which we relate one set to another, preserving or transforming structure. They are the morphisms of the category of sets, and understanding their properties is essential for grasping homomorphisms in algebra.

Definition

A **function** (or **map**) $f : A \to B$ from set A to set B is a rule that assigns to each element $a \in A$ exactly one element $f(a) \in B$. We call A the **domain**, B the **codomain**, and the set $f(A) = \{f(a) : a \in A\}$ the **image** or **range** of f.

Definition

Let $f : A \to B$ be a function.

- f is **injective** (or **one-to-one**) if $f(a_1) = f(a_2)$ implies $a_1 = a_2$ for all $a_1, a_2 \in A$.
- f is **surjective** (or **onto**) if for every $b \in B$, there exists $a \in A$ such that $f(a) = b$.
- f is **bijective** if it is both injective and surjective.

Theorem

A function $f : A \to B$ is bijective if and only if there exists a function $g : B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. Such a function g is unique and is called the **inverse** of f, denoted f^{-1}.

Proof. ($\Rightarrow$) Suppose f is bijective. For each $b \in B$, since f is surjective, there exists $a \in A$ with $f(a) = b$. This a is unique by injectivity. Define $g(b) = a$. Then for any $a \in A$, $g(f(a)) = a$, so $g \circ f = \mathrm{id}_A$. For any $b \in B$, if $g(b) = a$, then $f(a) = b$ by construction, so $f \circ g = \mathrm{id}_B$. ($\Leftarrow$) Suppose such g exists. For injectivity: if $f(a_1) = f(a_2)$, then $a_1 = g(f(a_1)) = g(f(a_2)) = a_2$. For surjectivity: given $b \in B$, let $a = g(b)$. Then $f(a) = f(g(b)) = b$. ∎

Binary Relations

Relations generalize the notion of order and equality, providing a framework for comparing and classifying elements within a set.

Definition

A **binary relation** R on a set A is a subset $R \subseteq A \times A$. We write aRb or $a \sim b$ to mean $(a, b) \in R$. A relation R is:

- **Reflexive** if aRa for all $a \in A$.
- **Symmetric** if aRb implies bRa for all $a, b \in A$.
- **Antisymmetric** if aRb and bRa imply $a = b$ for all $a, b \in A$.
- **Transitive** if aRb and bRc imply aRc for all $a, b, c \in A$.

Definition

An **equivalence relation** on A is a relation that is reflexive, symmetric, and transitive. For an equivalence relation $\sim$ on A and an element $a \in A$, the **equivalence class** of a is

$$[a] = \{b \in A : b \sim a\}$$

Theorem

Let $\sim$ be an equivalence relation on a set A. Then the set of equivalence classes $\{[a] : a \in A\}$ forms a partition of A. Conversely, any partition of A determines an equivalence relation on A.

> ***Proof.*** For the first direction, we must show: (1) every element of A belongs to some equivalence class, and (2) distinct equivalence classes are disjoint. For (1), every $a \in A$ belongs to $[a]$ by reflexivity. For (2), suppose $[a] \cap [b] \neq \emptyset$. Then there exists $c \in [a] \cap [b]$, so $c \sim a$ and $c \sim b$. By symmetry and transitivity, $a \sim b$. We claim $[a] = [b]$. If $x \in [a]$, then $x \sim a$, and since $a \sim b$, we have $x \sim b$ by transitivity, so $x \in [b]$. Similarly, $[b] \subseteq [a]$. For the converse, given a partition $\{A_i\}$ of A, define $a \sim b$ if a and b belong to the same partition element. This is easily verified to be an equivalence relation. ∎

> **Definition**
>
> A **partial order** on A is a relation that is reflexive, antisymmetric, and transitive. A set with a partial order is called a **partially ordered set** or **poset**. A partial order is a **total order** if for all $a, b \in A$, either $a \leq b$ or $b \leq a$.

1.3 Modular Arithmetic

Modular arithmetic, sometimes called "clock arithmetic," provides our first glimpse of a finite algebraic structure that will later be formalized as the cyclic group $\mathbb{Z}_n$.

> **Definition**
>
> Let $n \in \mathbb{Z}^+$ with $n > 1$. For $a, b \in \mathbb{Z}$, we say a is **congruent to** b **modulo** n, written $a \equiv b \pmod{n}$, if n divides $a - b$. That is, $a \equiv b \pmod{n}$ if and only if $a - b = kn$ for some $k \in \mathbb{Z}$.

> **Proposition**
>
> Congruence modulo n is an equivalence relation on $\mathbb{Z}$.

> ***Proof.*** Reflexivity: $a - a = 0 = 0 \cdot n$, so $a \equiv a \pmod{n}$. Symmetry: If $a \equiv b \pmod{n}$, then $a - b = kn$, so $b - a = -kn = (-k)n$, thus $b \equiv a \pmod{n}$. Transitivity: If $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, then $a - b = k_1 n$ and $b - c = k_2 n$, so $a - c = (a - b) + (b - c) = k_1 n + k_2 n = (k_1 + k_2)n$, thus $a \equiv c \pmod{n}$. ∎

The equivalence classes under this relation are called **residue classes** or **congruence classes** modulo n. The set of all such classes is denoted $\mathbb{Z}_n = \{[0], [1], [2], \ldots, [n-1]\}$.

> **Theorem**
>
> The operations $[a] + [b] = [a + b]$ and $[a] \cdot [b] = [ab]$ on $\mathbb{Z}_n$ are well-defined. That is, if $[a] = [a']$ and $[b] = [b']$, then $[a + b] = [a' + b']$ and $[ab] = [a'b']$.

Proof. Suppose $[a] = [a']$ and $[b] = [b']$. Then $a \equiv a' \pmod{n}$ and $b \equiv b' \pmod{n}$, so $a - a' = kn$ and $b - b' = \ell n$ for some $k, \ell \in \mathbb{Z}$. Then $(a+b) - (a'+b') = (a-a') + (b-b') = kn + \ell n = (k+\ell)n$, so $a + b \equiv a' + b' \pmod{n}$, i.e., $[a+b] = [a'+b']$. For multiplication, $ab - a'b' = ab - a'b + a'b - a'b' = (a-a')b + a'(b-b') = knb + a'\ell n = (kb + a'\ell)n$, so $ab \equiv a'b' \pmod{n}$, i.e., $[ab] = [a'b']$. ∎

Theorem

(Chinese Remainder Theorem) Let $n_1, n_2, \ldots, n_k$ be pairwise coprime positive integers, and let $n = n_1 n_2 \cdots n_k$. Then for any integers $a_1, a_2, \ldots, a_k$, the system of congruences

$$\begin{aligned} x &\equiv a_1 \pmod{n_1} \quad (3)\\ x &\equiv a_2 \pmod{n_2} \quad (4)\\ &\vdots \quad (5)\\ x &\equiv a_k \pmod{n_k} \quad (6) \end{aligned}$$

has a unique solution modulo n.

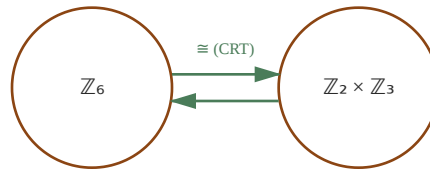


Figure 1.1: The Chinese Remainder Theorem establishes an isomorphism between $\mathbb{Z}_n$ and the product of coprime factors.

1.4 Proof Techniques

The art of proof is the heart of mathematics. While calculation verifies particular cases, only proof establishes universal truth. We examine the principal methods of mathematical demonstration.

Direct Proof

A direct proof proceeds from hypotheses to conclusion through a chain of logical implications.

> **Example**
>
> **Claim:** If n is an even integer, then n^2 is even.
>
> **Proof:** Since n is even, we can write $n = 2k$ for some integer k. Then $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$. Since $2k^2$ is an integer, n^2 is even.

Proof by Contradiction

To prove a statement P, we assume $\neg P$ and derive a logical contradiction, thereby establishing that P must be true.

> **Example**
>
> **Claim:** $\sqrt{2}$ is irrational.
>
> **Proof:** Suppose, for contradiction, that $\sqrt{2}$ is rational. Then $\sqrt{2} = \frac{p}{q}$ where $p, q \in \mathbb{Z}$, $q \neq 0$, and $\gcd(p, q) = 1$ (the fraction is in lowest terms). Squaring both sides: $2 = \frac{p^2}{q^2}$, so $p^2 = 2q^2$. Thus p^2 is even, which implies p is even (as we can verify: if p were odd, p^2 would be odd). Write $p = 2k$. Then $p^2 = 4k^2 = 2q^2$, so $q^2 = 2k^2$, making q^2 even, hence q even. But now both p and q are even, contradicting $\gcd(p, q) = 1$. This contradiction establishes that $\sqrt{2}$ is irrational.

Mathematical Induction

To prove that a property $P(n)$ holds for all natural numbers $n \geq n_0$, we proceed in two steps:

- **Base Case:** Prove $P(n_0)$.
- **Inductive Step:** Prove that $P(k) \Rightarrow P(k+1)$ for all $k \geq n_0$.

> **Theorem**
>
> For all $n \in \mathbb{N}$, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

> ***Proof.*** **Base case:** For $n = 1$, $\sum_{i=1}^{1} i = 1 = \frac{1(1+1)}{2}$. ✓ **Inductive step:** Assume $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$ for some $k \geq 1$. Then
>
> $$\sum_{i=1}^{k+1} i = \left(\sum_{i=1}^{k} i\right) + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$
>
> This completes the induction. ∎

Contrapositive

To prove $P \Rightarrow Q$, we may instead prove $\neg Q \Rightarrow \neg P$, which is logically equivalent.

Example

Claim: If n^2 is even, then n is even.

Proof (by contrapositive): We prove: if n is odd, then n^2 is odd. If n is odd, write $n = 2k + 1$. Then $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is odd.

1.5 Number Theory Foundations

Number theory, "the queen of mathematics," provides essential tools and examples throughout algebra. We establish the fundamental results concerning divisibility and prime factorization.

Definition

Let $a, b \in \mathbb{Z}$ with $b \neq 0$. We say b **divides** a, written $b \mid a$, if there exists $c \in \mathbb{Z}$ such that $a = bc$. We call b a **divisor** or **factor** of a, and a a **multiple** of b.

Definition

An integer $p > 1$ is **prime** if its only positive divisors are 1 and p. An integer $n > 1$ that is not prime is called **composite**.

Definition

Let $a, b \in \mathbb{Z}$, not both zero. The **greatest common divisor** of a and b, denoted $\gcd(a, b)$, is the largest positive integer that divides both a and b.

Theorem

(Euclidean Algorithm) For $a, b \in \mathbb{Z}$ with $b > 0$, we can compute $\gcd(a, b)$ by the following recursive procedure: If $b = 0$, then $\gcd(a, b) = |a|$. Otherwise, $\gcd(a, b) = \gcd(b, a \bmod b)$.

Example

Compute $\gcd(252, 105)$:

$$\begin{aligned} \gcd(252, 105) &= \gcd(105, 42) \quad \text{since } 252 = 2 \cdot 105 + 42 && (7)\\ &= \gcd(42, 21) \quad \text{since } 105 = 2 \cdot 42 + 21 && (8)\\ &= \gcd(21, 0) \quad \text{since } 42 = 2 \cdot 21 + 0 && (9)\\ &= 21 && (10) \end{aligned}$$

Theorem

(Bézout's Identity) Let $a, b \in \mathbb{Z}$, not both zero, and let $d = \gcd(a, b)$. Then there exist integers x, y such that $ax + by = d$.

Proof. Consider the set $S = \{am + bn : m, n \in \mathbb{Z}, am + bn > 0\}$. This set is nonempty (it contains $|a|$ and $|b|$) and consists of positive integers, so by well-ordering, it has a least element $d = ax + by$. We claim $d = \gcd(a, b)$. First, we show $d \mid a$. By division algorithm, $a = qd + r$ with $0 \leq r < d$. Then $r = a - qd = a - q(ax + by) = a(1 - qx) + b(-qy)$. If $r > 0$, then $r \in S$ with $r < d$, contradicting minimality of d. Thus $r = 0$, so $d \mid a$. Similarly, $d \mid b$. Now suppose c is any common divisor of a and b. Then $c \mid ax$ and $c \mid by$, so $c \mid (ax + by) = d$. Therefore $c \leq d$, establishing that $d = \gcd(a, b)$. ∎

Theorem

(Fundamental Theorem of Arithmetic) Every integer $n > 1$ can be expressed uniquely (up to order) as a product of prime numbers:

$$n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$$

where $p_1 < p_2 < \cdots < p_k$ are primes and $e_i \geq 1$.

Proof. We prove existence by strong induction. For $n = 2$, we have $n = 2$, which is prime. Suppose every integer $2 \leq m < n$ can be expressed as a product of primes. If n is prime, we're done. Otherwise, $n = ab$ where $1 < a, b < n$. By inductive hypothesis, both a and b are products of primes, hence so is $n = ab$. For uniqueness, suppose $n = p_1 p_2 \cdots p_r = q_1 q_2 \cdots q_s$ where all p_i and q_j are prime. Then $p_1 \mid q_1 q_2 \cdots q_s$. By Euclid's lemma (if p is prime and $p \mid ab$, then $p \mid a$ or $p \mid b$), we have $p_1 \mid q_j$ for some j. Since q_j is prime, $p_1 = q_j$. Reorder so $q_j = q_1$. Cancel p_1 from both sides to get $p_2 \cdots p_r = q_2 \cdots q_s$. Continue inductively to establish uniqueness. ∎

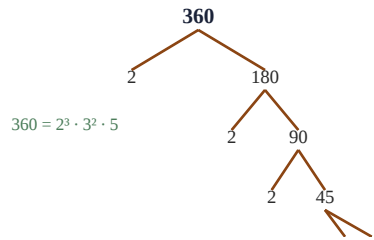


Figure 1.2: Prime factorization tree for 360.

❧

Greek Meditation

On the Eternal Forms of Number and the Platonic Realm

The ancient Greeks beheld in number and geometric form the eternal and unchanging reality beyond the flux of sensory experience. Plato's realm of Forms, those perfect and immutable templates of which material objects are but shadows, finds its clearest expression in mathematics. The number Three exists not in any particular collection of three objects, but as an eternal Form, accessible to reason alone.

When we prove that $\sqrt{2}$ is irrational, we do not merely discover a fact about symbols on a page—we glimpse an eternal truth, one that was true before humanity existed and will remain true long after. The prime numbers, in their infinite progression, follow laws we discover but do not create. We are, in the mathematician's highest calling, explorers of a realm more real than the physical world, archaeologists of eternal truth.

This is why proof matters: it is our means of ascent from the cave of particular examples to the sunlit realm of necessary and universal truth. Like the philosopher-mathematician ascending from shadows to Forms, we rise from computation to comprehension, from example to essence.

Zen Meditation

The Empty Set: Void as Foundation

Consider the empty set $\emptyset$, containing nothing, yet fundamental to all of mathematics. From emptiness, we build everything. The set $\{\emptyset\}$ contains one element: the void itself. From this we construct $\{\emptyset, \{\emptyset\}\}$, and so forth, building the natural numbers from pure absence.

This mirrors the Buddhist concept of Śūnyatā—emptiness not as mere negation, but as mathematical and super mathematical pleasurous potential, the source from which all forms arise. The empty set is not deficient but complete; it perfectly satisfies its definition, containing exactly what it should: nothing.

In Zen, before enlightenment: chop wood, carry water. After enlightenment: chop wood, carry water. Similarly, in mathematics, before understanding: manipulate symbols. After understanding: manipulate symbols. But the manipulation is transformed by comprehension, as mundane tasks are transformed by presence. The empty set remains empty, yet now we see it as the void from which infinite mathematics blossoms.

What is the sound of one hand clapping? What is the cardinality of the empty set? In both, the answer transcends and includes the question, pointing beyond itself to a deeper understanding.

Practice Problems

Complete solutions are provided below each problem.

1. Prove that for any sets A, B, and C: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

> ***Solution.*** Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. The latter means $x \in B$ or $x \in C$. Case 1: If $x \in B$, then $x \in A$ and $x \in B$, so $x \in A \cap B \subseteq (A \cap B) \cup (A \cap C)$. Case 2: If $x \in C$, then $x \in A$ and $x \in C$, so $x \in A \cap C \subseteq (A \cap B) \cup (A \cap C)$. Thus $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$. Conversely, let $x \in (A \cap B) \cup (A \cap C)$. Then $x \in A \cap B$ or $x \in A \cap C$. In either case, $x \in A$. Also, $x \in B$ or $x \in C$, so $x \in B \cup C$. Therefore $x \in A \cap (B \cup C)$, establishing $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

2. Show that $\mathbb{Q}$ (the rationals) is countable.

> ***Solution.*** We construct a bijection from $\mathbb{N}$ to $\mathbb{Q}^+$ (positive rationals); the extension to all rationals is straightforward. List all positive rationals in a grid where row n contains fractions with denominator n:
> Row 1: $1/1, 2/1, 3/1, 4/1, \ldots$
> Row 2: $1/2, 2/2, 3/2, 4/2, \ldots$
> Row 3: $1/3, 2/3, 3/3, 4/3, \ldots$
> Traverse diagonally: $1/1, 2/1, 1/2, 1/3, 2/2, 3/1, 4/1, 3/2, 2/3, 1/4, \ldots$, skipping any fraction already visited (like $2/2 = 1/1$). This enumerates $\mathbb{Q}^+$. For all rationals, interleave: $0, 1, -1, 1/2, -1/2, 2, -2, \ldots$

3. Prove that the function $f : \mathbb{R} \to (0, 1)$ defined by $f(x) = \frac{1}{1+e^{-x}}$ is bijective.

> ***Solution.*** **Injective:** Suppose $f(x_1) = f(x_2)$. Then $\frac{1}{1+e^{-x_1}} = \frac{1}{1+e^{-x_2}}$, so $1 + e^{-x_1} = 1 + e^{-x_2}$, thus $e^{-x_1} = e^{-x_2}$. Taking natural log: $-x_1 = -x_2$, so $x_1 = x_2$. ✓ **Surjective:** Let $y \in (0, 1)$. We seek x with $f(x) = y$. From $\frac{1}{1+e^{-x}} = y$, we get $1 = y(1 + e^{-x})$, so $1 = y + ye^{-x}$, thus $ye^{-x} = 1 - y$, giving $e^{-x} = \frac{1-y}{y}$. Take logs: $-x = \ln\left(\frac{1-y}{y}\right)$, so $x = -\ln\left(\frac{1-y}{y}\right) = \ln\left(\frac{y}{1-y}\right)$. This is well-defined for all $y \in (0, 1)$. ✓

4. Find all solutions to the system: $x \equiv 2 \pmod 3$, $x \equiv 3 \pmod 5$, $x \equiv 4 \pmod 7$.

> ***Solution.*** By CRT, there is a unique solution modulo $3 \cdot 5 \cdot 7 = 105$. Let $N = 105$, $n_1 = 3$, $n_2 = 5$, $n_3 = 7$. Define $N_1 = N/n_1 = 35$, $N_2 = N/n_2 = 21$, $N_3 = N/n_3 = 15$. Find M_i such that $N_i M_i \equiv 1 \pmod{n_i}$: - $35M_1 \equiv 1 \pmod 3$: Since $35 \equiv 2 \pmod 3$, we need $2M_1 \equiv 1 \pmod 3$. Try $M_1 = 2$: $2 \cdot 2 = 4 \equiv 1 \pmod 3$. ✓ - $21M_2 \equiv 1 \pmod 5$: Since $21 \equiv 1 \pmod 5$, we have $M_2 = 1$. ✓ - $15M_3 \equiv 1 \pmod 7$: Since $15 \equiv 1 \pmod 7$, we have $M_3 = 1$. ✓ Solution:
> $x \equiv a_1N_1M_1 + a_2N_2M_2 + a_3N_3M_3 = 2(35)(2) + 3(21)(1) + 4(15)(1) = 140 + 63 + 60$
> . Therefore $x = 53 + 105k$ for $k \in \mathbb{Z}$.

5. Prove by induction that $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$ for all $n \geq 0$.

> ***Solution.*** **Base case ($n = 0$):** $\sum_{i=0}^{0} 2^i = 2^0 = 1$ and $2^{0+1} - 1 = 2 - 1 = 1$. ✓ **Inductive step:** Assume $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$ for some $k \geq 0$. Then:

$$\sum_{i=0}^{k+1} 2^i = \left(\sum_{i=0}^{k} 2^i\right) + 2^{k+1} = (2^{k+1} - 1) + 2^{k+1} = 2 \cdot 2^{k+1} - 1 = 2^{k+2} - 1$$

This completes the induction. ∎

6. Use the Euclidean algorithm to compute $\gcd(1071, 462)$ and find integers x, y such that $1071x + 462y = \gcd(1071, 462)$.

> ***Solution.*** **Euclidean algorithm:**
>
> $$1071 = 2 \cdot 462 + 147 \tag{11}$$
> $$462 = 3 \cdot 147 + 21 \tag{12}$$
> $$147 = 7 \cdot 21 + 0 \tag{13}$$
>
> Thus $\gcd(1071, 462) = 21$. **Extended Euclidean algorithm (working backwards):**
>
> $$21 = 462 - 3 \cdot 147 \tag{14}$$
> $$= 462 - 3(1071 - 2 \cdot 462) \tag{15}$$
> $$= 462 - 3 \cdot 1071 + 6 \cdot 462 \tag{16}$$
> $$= 7 \cdot 462 - 3 \cdot 1071 \tag{17}$$
> $$= 1071(-3) + 462(7) \tag{18}$$
>
> Therefore $x = -3$ and $y = 7$.

7. Prove that if $\gcd(a, b) = 1$ and $a \mid bc$, then $a \mid c$.

> ***Solution.*** Since $\gcd(a, b) = 1$, by Bézout's identity there exist integers x, y such that $ax + by = 1$. Multiply both sides by c: $acx + bcy = c$. Now, $a \mid acx$ (obviously) and $a \mid bcy$ (since $a \mid bc$ by hypothesis, we can write $bc = ak$ for some integer k, so $bcy = aky$, which is divisible by a). Therefore a divides their sum: $a \mid (acx + bcy) = c$. ∎

8. Prove that there are infinitely many prime numbers.

> ***Solution.*** (Euclid's proof) Suppose, for contradiction, that there are only finitely many primes $p_1, p_2, \ldots, p_n$. Consider $N = p_1 p_2 \cdots p_n + 1$. By the Fundamental Theorem of Arithmetic, N has a prime divisor p. This prime p must be one of $p_1, p_2, \ldots, p_n$ (since we assumed these are all primes). Thus $p \mid N$ and $p \mid p_1 p_2 \cdots p_n$, so $p \mid (N - p_1 p_2 \cdots p_n) = 1$. But no prime divides 1, contradiction. Therefore there are infinitely many primes. ∎

9. Show that the composition of two bijections is a bijection.

> ***Solution.*** Let $f : A \to B$ and $g : B \to C$ be bijections. We show $g \circ f : A \to C$ is bijective. **Injective:** Suppose $(g \circ f)(a_1) = (g \circ f)(a_2)$. Then $g(f(a_1)) = g(f(a_2))$. Since g is injective, $f(a_1) = f(a_2)$. Since f is injective, $a_1 = a_2$. ✓ **Surjective:** Let $c \in C$. Since g is surjective, there exists $b \in B$ with $g(b) = c$. Since f is surjective, there exists $a \in A$ with $f(a) = b$. Then $(g \circ f)(a) = g(f(a)) = g(b) = c$. ✓ Therefore $g \circ f$ is bijective. ∎

10. Let $\sim$ be an equivalence relation on A with exactly 3 equivalence classes. If $|A| = 10$, what are the possible sizes of the equivalence classes?

Solution. The equivalence classes partition A, so their sizes sum to 10. We need three positive integers summing to 10. The possibilities are:

- $(8, 1, 1)$ and permutations: classes of sizes 8, 1, 1
- $(7, 2, 1)$ and permutations: classes of sizes 7, 2, 1
- $(6, 3, 1)$ and permutations: classes of sizes 6, 3, 1
- $(6, 2, 2)$ and permutations: classes of sizes 6, 2, 2
- $(5, 4, 1)$ and permutations: classes of sizes 5, 4, 1
- $(5, 3, 2)$ and permutations: classes of sizes 5, 3, 2
- $(4, 4, 2)$ and permutations: classes of sizes 4, 4, 2
- $(4, 3, 3)$ and permutations: classes of sizes 4, 3, 3

End of Chapter 1

Chapter 2: Introduction to Groups

❦

The concept of a group crystallizes the mathematical essence of symmetry. From the rotations of a crystal to the permutations of a Rubik's cube, from the integers under addition to the transformations of space itself, groups capture the fundamental structure underlying all symmetric phenomena.

2.1 Group Axioms and Definition

Definition

A **group** is an ordered pair $(G, \cdot)$ consisting of a nonempty set G and a binary operation $\cdot : G \times G \to G$ satisfying:

1. **Associativity:** $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in G$.
2. **Identity:** There exists $e \in G$ such that $e \cdot a = a \cdot e = a$ for all $a \in G$.
3. **Inverses:** For each $a \in G$, there exists $b \in G$ such that $a \cdot b = b \cdot a = e$.

Theorem

In any group $(G, \cdot)$:

1. The identity element is unique.
2. Each element has a unique inverse, denoted a^{-1}.
3. $(a^{-1})^{-1} = a$ for all $a \in G$.
4. $(ab)^{-1} = b^{-1}a^{-1}$ for all $a, b \in G$.

> ***Proof.*** (1) If e and e' are identities, then $e = ee' = e'$. (2) If b and c are inverses of a, then $b = be = b(ac) = (ba)c = ec = c$. (3) Since a inverts a^{-1}, we have $(a^{-1})^{-1} = a$. (4) $(ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1} = aea^{-1} = aa^{-1} = e$, similarly $(b^{-1}a^{-1})(ab) = e$. ∎

Definition

(Abelian Group) A group $(G, \cdot)$ is called **abelian** (or **commutative**) if the group operation satisfies the commutative law:

$$a \cdot b = b \cdot a$$

for all $a, b \in G$. A group that is not abelian is called **nonabelian**.

The term honors the Norwegian mathematician Niels Henrik Abel (1802–1829), whose pioneering work on polynomial equations and elliptic functions revealed the fundamental importance of commutativity in algebraic structures. Abel proved that there is no general formula for solving polynomial equations of degree five or higher—a result intimately connected to the non-abelian nature of certain symmetry groups.

Example

Abelian groups:

- $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$, $(\mathbb{C}, +)$ are all abelian since $a + b = b + a$.
- $(\mathbb{Z}_n, +)$ is abelian for all $n \geq 1$.
- $(\mathbb{R}^*, \cdot)$ and $(\mathbb{C}^*, \cdot)$ (nonzero reals/complex numbers under multiplication) are abelian.
- The Klein four-group $V_4 = \{e, a, b, c\}$ with $a^2 = b^2 = c^2 = e$ and $ab = c$ is abelian.

Example

Nonabelian groups:

- S_n for $n \geq 3$ is nonabelian. For instance, in S_3: $(12)(13) = (132)$ but $(13)(12) = (123)$.
- D_n for $n \geq 3$ is nonabelian. Rotations and reflections do not generally commute.
- $GL_n(\mathbb{R})$ for $n \geq 2$ (invertible $n \times n$ matrices) is nonabelian since matrix multiplication is not commutative.

2.2 First Examples

Example

Integers under addition: $(\mathbb{Z}, +)$ is an abelian group with identity 0 and inverse $-n$ for n.

Example

Modular arithmetic: $(\mathbb{Z}_n, +)$ is an abelian group of order n. For $n = 6$, elements are $\{[0], [1], [2], [3], [4], [5]\}$.

Example

Symmetric group S_3: The group of all permutations of $\{1, 2, 3\}$ has 6 elements and is nonabelian.

$\cdot$	e	(12)	(13)	(23)	(123)	(132)
e	e	(12)	(13)	(23)	(123)	(132)
(12)	(12)	e	(132)	(123)	(23)	(13)
(13)	(13)	(123)	e	(132)	(12)	(23)
(23)	(23)	(132)	(123)	e	(13)	(12)
(123)	(123)	(13)	(23)	(12)	(132)	e
(132)	(132)	(23)	(12)	(13)	e	(123)

2.3 Order of Elements

Definition

The **order** of an element a in a group G, denoted $|a|$ or $\text{ord}(a)$, is the smallest positive integer n such that $a^n = e$. If no such n exists, we say a has infinite order.

Theorem

Let a be an element of group G with $|a| = n < \infty$. Then:

1. $a^k = e$ if and only if $n \mid k$.
2. $|a^k| = n / \gcd(n, k)$.
3. The elements $e, a, a^2, \ldots, a^{n-1}$ are all distinct.

2.4 Cyclic Groups

Definition

A group G is **cyclic** if there exists $a \in G$ such that $G = \langle a \rangle = \{a^n : n \in \mathbb{Z}\}$. We call a a **generator** of G.

Theorem

(Structure of Cyclic Groups) Every cyclic group is isomorphic to either $(\mathbb{Z}, +)$ or $(\mathbb{Z}_n, +)$ for some n.

Proof. Let $G = \langle a \rangle$. Define $\varphi : \mathbb{Z} \to G$ by $\varphi(n) = a^n$. This is a surjective homomorphism. If $\ker(\varphi) = \{0\}$, then φ is an isomorphism and $G \cong \mathbb{Z}$. Otherwise, $\ker(\varphi) = n\mathbb{Z}$ for some $n > 0$, giving $G \cong \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$. ∎

2.5 Symmetric Groups

Definition

The **symmetric group** S_n is the group of all bijections $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ under composition. It has order $n!$.

Definition

A **cycle** of length k is a permutation $(a_1\ a_2\ \cdots\ a_k)$ that maps $a_1 \to a_2 \to \cdots \to a_k \to a_1$ and fixes all other elements. Cycles are **disjoint** if they move no common elements.

Theorem

Every permutation in S_n can be written uniquely (up to order) as a product of disjoint cycles.

2.5.1 Two-Line Notation

A permutation $\sigma \in S_n$ can be represented explicitly by showing where each element maps. The **two-line notation** displays the domain elements in the first row and their images in the second:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \sigma(1) & \sigma(2) & \sigma(3) & \cdots & \sigma(n) \end{pmatrix}$$

Example

The permutation that sends $1 \mapsto 3$, $2 \mapsto 1$, $3 \mapsto 4$, $4 \mapsto 2$ is written:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$$

To compute σ^2, we trace two applications: $1 \xrightarrow{\sigma} 3 \xrightarrow{\sigma} 4$, $2 \xrightarrow{\sigma} 1 \xrightarrow{\sigma} 3$, etc., giving:

$$\sigma^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

While explicit, two-line notation becomes cumbersome for large n and obscures the cycle structure. Cycle notation provides a more compact and revealing representation.

2.5.2 Cycle Notation

A k-**cycle** (or cycle of length k) is written $(a_1\ a_2\ \cdots\ a_k)$ and represents the permutation:

$$a_1 \to a_2 \to a_3 \to \cdots \to a_k \to a_1$$

while fixing all elements not appearing in the cycle. A 2-cycle is called a **transposition**.

Example

- The cycle $(1\ 3\ 4)$ sends $1 \to 3$, $3 \to 4$, $4 \to 1$, and fixes 2.
- The permutation from the previous example is the single cycle $(1\ 3\ 4\ 2)$.
- The identity permutation can be written as (1) or simply e.
- $(1\ 3\ 5)(2\ 4)$ is a product of disjoint cycles in S_5.

Cycle notation is not unique: $(1\ 3\ 4\ 2) = (3\ 4\ 2\ 1) = (4\ 2\ 1\ 3) = (2\ 1\ 3\ 4)$. By convention, we often start with the smallest element, though this is not required.

Definition

Two cycles are **disjoint** if they have no elements in common. Disjoint cycles commute: if α and β are disjoint, then $\alpha\beta = \beta\alpha$.

> ***Proof.*** Let α and β be disjoint cycles. For any element i:
>
> - • If i is moved by α but not β: $(\alpha\beta)(i) = \alpha(i) = (\beta\alpha)(i)$.
> - • If i is moved by β but not α: $(\alpha\beta)(i) = \beta(i) = (\beta\alpha)(i)$.
> - • If i is fixed by both: $(\alpha\beta)(i) = i = (\beta\alpha)(i)$.
>
> Thus $\alpha\beta = \beta\alpha$. ∎

Convention: Composition is read *right to left*. Thus $(1\ 2)(2\ 3)$ means "first apply $(2\ 3)$, then $(1\ 2)$," giving $1 \to 1 \to 2$, $2 \to 3 \to 3$, $3 \to 2 \to 1$, which equals $(1\ 2\ 3)$.

2.5.3 Transpositions and Cycle Decomposition

> **Theorem**
>
> Every cycle can be written as a product of transpositions. Specifically:
>
> $$(a_1\ a_2\ \cdots\ a_k) = (a_1\ a_k)(a_1\ a_{k-1})\cdots(a_1\ a_3)(a_1\ a_2)$$

> ***Proof.*** We verify the right side produces the correct mapping. Applying the transpositions right to left:
>
> - • $a_1 \xrightarrow{(a_1 a_2)} a_2 \xrightarrow{\text{fixed}} \cdots \xrightarrow{\text{fixed}} a_2$
> - • $a_2 \xrightarrow{(a_1 a_2)} a_1 \xrightarrow{(a_1 a_3)} a_3 \xrightarrow{\text{fixed}} \cdots \xrightarrow{\text{fixed}} a_3$
> - • $a_i \xrightarrow{\text{fixed}} \cdots \xrightarrow{(a_1 a_i)} a_1 \xrightarrow{(a_1 a_{i+1})} a_{i+1} \xrightarrow{\text{fixed}} \cdots \xrightarrow{\text{fixed}} a_{i+1}$
> - • $a_k \xrightarrow{\text{fixed}} \cdots \xrightarrow{(a_1 a_k)} a_1$
>
> This matches the cycle $(a_1\ a_2\ \cdots\ a_k)$. ∎

> **Example**
>
> Express $(1\ 3\ 4\ 2)$ as a product of transpositions:
>
> $$(1\ 3\ 4\ 2) = (1\ 2)(1\ 4)(1\ 3)$$
>
> We can verify: $1 \xrightarrow{(1\ 3)} 3 \xrightarrow{(1\ 4)} 3 \xrightarrow{(1\ 2)} 3$ ✓, $3 \xrightarrow{(1\ 3)} 1 \xrightarrow{(1\ 4)} 4 \xrightarrow{(1\ 2)} 4$ ✓, etc.

> **Theorem**
>
> **(Transposition Decomposition)** Every permutation in S_n can be written as a product of transpositions.

> ***Proof.*** By the cycle decomposition theorem, every permutation is a product of disjoint cycles. By the previous theorem, each cycle can be written as a product of transpositions. Composing these gives the result. ∎

The decomposition into transpositions is *not unique*. For example, $(1\ 2\ 3) = (1\ 3)(1\ 2) = (2\ 3)(1\ 3) = (1\ 2)(2\ 3)(1\ 2)(2\ 3)$. However, the *parity* of the number of transpositions is invariant.

2.5.4 Even and Odd Permutations

> **Definition**
>
> A permutation is **even** if it can be written as a product of an even number of transpositions, and **odd** if it requires an odd number. The **sign** (or **signature**) of σ is:
>
> $$\operatorname{sgn}(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases}$$

> **Theorem**
>
> The sign function is well-defined: no permutation can be both even and odd. Moreover, $\operatorname{sgn} : S_n \to \{\pm 1\}$ is a group homomorphism.

Proof sketch: Define $\operatorname{sgn}(\sigma) = \prod_{i<j} \frac{\sigma(i)-\sigma(j)}{i-j}$. This equals ± 1 and satisfies $\operatorname{sgn}(\sigma\tau) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)$, making it a homomorphism. Since $\operatorname{sgn}((i\ j)) = -1$, the sign counts transposition parity.

> **Example**
>
> - All transpositions are odd: $\operatorname{sgn}((i\ j)) = -1$.
> - $(1\ 2\ 3) = (1\ 3)(1\ 2)$ is even (product of 2 transpositions).
> - $(1\ 2\ 3\ 4) = (1\ 4)(1\ 3)(1\ 2)$ is odd (product of 3 transpositions).
> - In general, a k-cycle is even if k is odd, and odd if k is even.

> **Definition**
>
> The **alternating group** $A_n = \{\sigma \in S_n : \operatorname{sgn}(\sigma) = 1\}$ is the subgroup of even permutations. It has order $|A_n| = n!/2$.

The alternating groups are among the most important in mathematics. A_5 is the smallest nonabelian simple group (no nontrivial normal subgroups), and for $n \geq 5$, A_n is simple. This simplicity is intimately connected to the insolvability of the quintic.

2.6 Dihedral Groups: Symmetries of Regular Polygons

Among the most geometrically intuitive groups are the dihedral groups, which encode the symmetries of regular polygons. These groups bridge the abstract algebraic world with concrete geometric transformations, providing a rich playground for group-theoretic exploration.

2.6.1 Geometric Definition

> **Definition**
>
> The **dihedral group** D_n is the group of symmetries (distance-preserving transformations) of a regular n-gon. It consists of:
>
> - n rotations about the center: $0^\circ, 360^\circ/n, 2 \cdot 360^\circ/n, \ldots, (n-1) \cdot 360^\circ/n$
> - n reflections through axes of symmetry
>
> Thus $|D_n| = 2n$.

> **Example**
>
> **D_3: Symmetries of an Equilateral Triangle**
> The group D_3 has 6 elements:
>
> - **Rotations:** $r_0 = e$ (identity), r_1 (rotation by 120°), r_2 (rotation by 240°)
> - **Reflections:** s_1, s_2, s_3 through the three axes connecting vertices to opposite edge midpoints
>
> Note that $D_3 \cong S_3$. The correspondence arises because rotating or reflecting a triangle permutes its three vertices, giving an isomorphism $D_3 \to S_3$.

> **Example**
>
> **D_4: Symmetries of a Square**
> The group D_4 has 8 elements:
>
> - **Rotations:** e, r, r^2, r^3 (by $0^\circ, 90^\circ, 180^\circ, 270^\circ$)
> - **Reflections:** Two through opposite sides (horizontal and vertical), two through opposite vertices (diagonals)
>
> Unlike D_3, we have $D_4 \ncong S_4$ (since $|D_4| = 8 \neq 24 = |S_4|$).

For $n = 2$, we obtain the **Klein four-group**: $D_2 = \{e, r, s_1, s_2\}$ with r a 180° rotation and s_1, s_2 perpendicular reflections. This is the symmetry group of a (non-square) rectangle.

2.6.2 Algebraic Presentation

The geometric structure of D_n admits a concise algebraic description through **generators and relations**.

> **Definition**
>
> The dihedral group D_n has the presentation:
>
> $$D_n = \langle r, s \mid r^n = s^2 = e,\ srs = r^{-1} \rangle$$
>
> where:
>
> - r represents rotation by $360^\circ/n$ (the minimal rotation)
> - s represents any reflection
> - The relation $r^n = e$ captures that n rotations return to identity
> - The relation $s^2 = e$ says reflections are involutions (self-inverse)
> - The relation $srs = r^{-1}$ (equivalently $sr = r^{-1}s$) encodes how reflections reverse rotation direction

This presentation allows us to write every element of D_n uniquely in the form r^i or sr^i for $0 \leq i < n$. The rotations form the **cyclic subgroup** $\langle r \rangle = \{e, r, r^2, \ldots, r^{n-1}\}$ of order n, and the reflections are the n elements $\{s, sr, sr^2, \ldots, sr^{n-1}\}$.

> **Example**
>
> In D_4, using r (rotation by 90°) and s (reflection), we can compute:
>
> $sr \cdot sr^2 = s(rs)r^2 = s(r^{-1}s^{-1})r^2 = sr^{-1}s^{-1}r^2 = sr^3sr^2 = sr^3s^{-1}r^2 = sr^{-3}r^2 = sr^{-1} = sr^3$
>
> using the relations $sr = r^{-1}s$ and $s^2 = e$.

2.6.3 Properties of Dihedral Groups

Theorem

For all $n \geq 3$:

1. D_n is nonabelian.
2. The rotations $\{e, r, r^2, \ldots, r^{n-1}\}$ form a cyclic (hence abelian) normal subgroup of index 2.
3. Every reflection has order 2.
4. $Z(D_n) = \begin{cases} \{e\} & \text{if } n \text{ is odd} \\ \{e, r^{n/2}\} & \text{if } n \text{ is even} \end{cases}$

Proof. (1) Since $sr \neq rs$ (as $sr = r^{-1}s$), D_n is nonabelian for $n \geq 3$. (2) The rotations clearly form a subgroup isomorphic to $\mathbb{Z}_n$. It's normal since any reflection conjugates a rotation to its inverse: $sr^i s^{-1} = sr^i s = r^{-i}$. (3) $s^2 = e$ by the presentation. (4) For n odd, no rotation except e commutes with all reflections. For n even, the half-turn $r^{n/2}$ is the unique nontrivial central element. ∎

Proposition

The conjugacy classes of D_n are:

- $\{e\}$
- $\{r^k, r^{-k}\}$ for $1 \leq k < n/2$ (pairs of inverse rotations)
- $\{r^{n/2}\}$ if n is even (the half-turn)
- Conjugacy classes of reflections (structure depends on parity of n)

For n odd, all reflections form a single conjugacy class of size n. For n even, reflections split into two conjugacy classes: those through opposite vertices and those through opposite edge midpoints.

2.6.4 Applications and Significance

Dihedral groups appear throughout mathematics, science, and art:

Chemistry: Molecular symmetry analysis uses dihedral groups extensively. Benzene (C_6H_6) has D_6 symmetry, ferrocene exhibits D_5 symmetry, and many organic molecules display dihedral symmetries. These symmetries constrain molecular orbitals and predict spectroscopic properties.

Crystallography: Two-dimensional crystal patterns (wallpaper groups) contain dihedral subgroups. The 17 wallpaper groups encode all possible periodic tilings of the plane, with dihedral symmetries playing a central role.

Art and Architecture: Islamic geometric art extensively employs patterns with D_4, D_6, D_8, and D_{12} symmetries. Mandalas in Buddhist and Hindu traditions often exhibit D_4 or D_8 symmetry. Gothic rose windows display dihedral symmetries ranging from D_6 to D_{12}.

Robotics and Computer Graphics: Motion planning for symmetric objects exploits dihedral symmetry to reduce computational complexity. Rendering algorithms use symmetry groups to optimize calculations.

Music Theory: The neo-Riemannian theory of harmony uses D_{12} (symmetries of the 12-tone chromatic scale) to analyze chord transformations and tonal relationships.

> **Example**
>
> **Cayley Table for D_3:**
> Using the presentation $D_3 = \langle r, s \mid r^3 = s^2 = e, sr = r^{-1}s = r^2s \rangle$:
>
> | $\cdot$ | e | r | r^2 | s | sr | sr^2 |
> |---|---|---|---|---|---|---|
> | e | e | r | r^2 | s | sr | sr^2 |
> | r | r | r^2 | e | sr^2 | s | sr |
> | r^2 | r^2 | e | r | sr | sr^2 | s |
> | s | s | sr | sr^2 | e | r | r^2 |
> | sr | sr | sr^2 | s | r^2 | e | r |
> | sr^2 | sr^2 | s | sr | r | r^2 | e |
>
> Compare this with the Cayley table for S_3 given earlier—the isomorphism becomes evident!

2.7 Special Groups and Fundamental Theorems

Having explored cyclic groups, symmetric groups, and dihedral groups, we now examine two foundational topics: the Klein four-group (the simplest nonabelian—actually abelian!—non-cyclic group) and Cayley's theorem, which reveals that every abstract group is secretly a permutation group.

2.7.1 The Klein Four-Group

> **Definition**
>
> The **Klein four-group** (German: *Vierergruppe*), denoted V_4 or K_4, is the group of order 4 in which every non-identity element has order 2. It can be defined by the presentation:
>
> $$V_4 = \langle a, b \mid a^2 = b^2 = (ab)^2 = e \rangle$$
>
> Explicitly: $V_4 = \{e, a, b, ab\}$ with $ab = ba$.

Named after Felix Klein (1849–1925), this group is remarkable for being the smallest group that is both *abelian* and *non-cyclic*. Since every element has order 2, there is no single element that generates the entire group.

Example

Realizations of V_4:

- **Direct product:** $V_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0),(1,0),(0,1),(1,1)\}$ under component-wise addition mod 2.
- **Dihedral group:** $V_4 \cong D_2$, the symmetry group of a non-square rectangle (identity, 180° rotation, horizontal reflection, vertical reflection).
- **Symmetric group:** $V_4 \cong \{e, (12)(34), (13)(24), (14)(23)\} \subset S_4$, the subgroup of double transpositions.
- **Matrix group:** $V_4 \cong \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$ under matrix multiplication.

Theorem

There are exactly two groups of order 4 (up to isomorphism): the cyclic group $\mathbb{Z}_4$ and the Klein four-group V_4.

> ***Proof.*** Let G be a group of order 4. By Lagrange's theorem (to be proven later), every element has order dividing 4, i.e., order 1, 2, or 4. If G has an element of order 4, then $G \cong \mathbb{Z}_4$. Otherwise, every non-identity element has order 2. Pick $a, b \in G \setminus \{e\}$ with $a \neq b$. Then $ab \notin \{e, a, b\}$ (else we'd have contradictions like $ab = a \Rightarrow b = e$). Thus $G = \{e, a, b, ab\}$. We compute $(ab)^2 = a(ba)b$. If $ba = ab$, then $(ab)^2 = a^2b^2 = e$, giving the Klein four-group structure. If $ba \neq ab$, then... (one can show this leads to contradiction with $|G| = 4$). Thus $G \cong V_4$. ∎

The Klein four-group appears naturally in many contexts: as the Galois group of certain polynomials, in the classification of abelian groups, and in topology (as the fundamental group of the Klein bottle minus a point).

2.7.2 Cayley's Theorem

One of the most philosophically profound results in group theory is that *every group is a group of permutations*. This bridges the abstract definition of groups with the concrete symmetric groups.

Theorem

(Cayley's Theorem, 1854) Every group G is isomorphic to a subgroup of the symmetric group $S_{|G|}$ on $|G|$ elements.

Proof. We construct an injection $\varphi : G \to S_G$ (where S_G is the group of permutations of the set G). For each $g \in G$, define the **left multiplication map** $\lambda_g : G \to G$ by $\lambda_g(x) = gx$. This is a bijection: it's injective since $gx = gy \Rightarrow x = y$ (cancel g), and surjective since for any $y \in G$, we have $\lambda_g(g^{-1}y) = y$.

Define $\varphi : G \to S_G$ by $\varphi(g) = \lambda_g$. We verify this is a homomorphism:

$$\varphi(gh)(x) = \lambda_{gh}(x) = (gh)x = g(hx) = \lambda_g(\lambda_h(x)) = (\lambda_g \circ \lambda_h)(x) = (\varphi(g) \circ \varphi(h))(x)$$

Thus $\varphi(gh) = \varphi(g)\varphi(h)$. Moreover, φ is injective: if $\varphi(g) = \varphi(h)$, then $\lambda_g = \lambda_h$, so $\lambda_g(e) = \lambda_h(e)$, giving $g = h$. Therefore $G \cong \varphi(G) \leq S_G$. Since $|G| = |\text{underlying set}|$, we have $S_G \cong S_{|G|}$. ∎

Example

Cayley representation of $\mathbb{Z}_3$:
Consider $\mathbb{Z}_3 = \{0, 1, 2\}$ under addition mod 3. The left multiplication maps are:

- λ_0: identity permutation (fixes everything)
- λ_1: $0 \mapsto 1, 1 \mapsto 2, 2 \mapsto 0$, i.e., the cycle $(0\ 1\ 2)$
- λ_2: $0 \mapsto 2, 1 \mapsto 0, 2 \mapsto 1$, i.e., the cycle $(0\ 2\ 1)$

Thus $\mathbb{Z}_3 \cong \{e, (0\ 1\ 2), (0\ 2\ 1)\} \leq S_3$.

Cayley's theorem is simultaneously trivial (the proof is straightforward) and profound (it completely characterizes what groups "are"). It shows that the study of abstract groups is, in principle, reducible to the study of permutation groups. However, the representation given by Cayley's theorem is often neither minimal nor particularly illuminating—finding "natural" or "small" permutation representations remains an art.

Example

Why Cayley's representation may be inefficient: The dihedral group D_n has order $2n$, so Cayley's theorem embeds it in S_{2n}. But D_n acts naturally on the n vertices of an n-gon, giving a much more efficient embedding into S_n.

2.7.3 Historical Note: The Genesis of Group Theory

Historical Meditation: From Equations to Axioms

The concept of a group emerged gradually through the 19th century, not from abstract contemplation, but from the desperate struggle to solve polynomial equations. For millennia, mathematicians sought formulas for solving equations:

- **Quadratic ($ax^2 + bx + c = 0$):** Known to ancient Babylonians (~2000 BCE)
- **Cubic ($x^3 + px + q = 0$):** Solved by Cardano and Tartaglia (1535)
- **Quartic ($x^4 + \cdots = 0$):** Solved by Ferrari (1540)
- **Quintic ($x^5 + \cdots = 0$):** ???

For centuries, the quintic resisted all attempts. Then, in the 1820s, two young geniuses provided the answer—but not the answer anyone expected. Niels Henrik Abel (1802–1829) proved that there is *no general formula* for solving quintic equations using radicals. Évariste Galois (1811–1832), writing feverishly the night before his fatal duel at age 20, went further: he characterized *exactly which* polynomials can be solved by radicals.

Galois's revolutionary insight was to study the *symmetries of roots* rather than the roots themselves. These symmetries form what we now call the Galois group of the polynomial. The polynomial is solvable by radicals if and only if its Galois group is "solvable" (a technical condition involving normal subgroups). For the general quintic, the Galois group is S_5, which is not solvable because its derived series terminates at the simple group A_5.

Thus, from the concrete problem of solving equations emerged the abstract notion of a group. Arthur Cayley gave the first formal definition in 1854, recognizing that these "groups of substitutions" (permutations) formed a fundamental mathematical structure. Later, the definition was abstracted further to encompass any set with an associative operation, identity, and inverses.

The journey from quadratic formulas to abstract algebra exemplifies a profound pattern in mathematics: *Problems of calculation lead to theories of structure.* We sought to compute roots; we discovered symmetry. We asked "how?"; mathematics answered with "why?"

Greek Meditation: Symmetry as Arche

The Pre-Socratics sought the *arche*—the fundamental principle from which all reality springs. For Thales it was water, for Anaximenes air, for Heraclitus fire and flux. But perhaps the deepest *arche* is symmetry itself. A snowflake's six-fold rotational symmetry mirrors the group C_6. The bilateral symmetry of the human form embodies the dihedral group. In uncovering group structure, we glimpse the eternal patterns—the Platonic Forms—that underlie the cosmos.

Zen Meditation: Identity and Transformation

The identity element is both nothing and everything: it acts yet changes nothing. Like the Zen concept of *mu* (emptiness), it is the still point around which all transformations dance.

> When you compose an element with its inverse, difference collapses into sameness. The journey out returns home. *Before enlightenment, chop wood carry water. After enlightenment, chop wood carry water.* The actions appear identical, yet understanding has transformed them. So too with groups: the operation seems the same, but comprehension of structure reveals infinite depth in finite patterns.

Practice Problems

1. Prove that in any group, $(ab)^{-1} = b^{-1}a^{-1}$.

> ***Solution.*** We have $(ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1} = aea^{-1} = aa^{-1} = e$. Similarly $(b^{-1}a^{-1})(ab) = e$. By uniqueness of inverses, $(ab)^{-1} = b^{-1}a^{-1}$. ∎

2. Find the order of each element in $\mathbb{Z}_{12}$.

> ***Solution.*** $|[a]| = 12/\gcd(12, a)$: $|[1]| = 12$, $|[2]| = 6$, $|[3]| = 4$, $|[4]| = 3$, $|[5]| = 12$, $|[6]| = 2$, $|[7]| = 12$, $|[8]| = 3$, $|[9]| = 4$, $|[10]| = 6$, $|[11]| = 12$. ∎

3. Show that S_3 is nonabelian.

> ***Solution.*** Consider (12) and (13). We have $(12)(13) = (132)$ but $(13)(12) = (123) \neq (132)$. ∎

4. Prove every subgroup of a cyclic group is cyclic.

> ***Solution.*** Let $G = \langle a \rangle$ and $H \leq G$. If $H = \{e\}$, done. Otherwise, let m be the smallest positive integer with $a^m \in H$. Claim: $H = \langle a^m \rangle$. Clearly $\langle a^m \rangle \subseteq H$. For the reverse, let $a^k \in H$. By division algorithm, $k = qm + r$ with $0 \leq r < m$. Then $a^r = a^k(a^m)^{-q} \in H$. By minimality of m, $r = 0$, so $a^k = (a^m)^q \in \langle a^m \rangle$. ∎

5. Determine the generators of $\mathbb{Z}_{20}$.

> ***Solution.*** $[a]$ generates $\mathbb{Z}_n$ iff $\gcd(a, n) = 1$. So generators of $\mathbb{Z}_{20}$ are: $[1], [3], [7], [9], [11], [13], [17], [19]$. There are $\varphi(20) = 8$ generators. ∎

6. Write the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 1 & 6 & 2 & 4 \end{pmatrix}$ as a product of disjoint cycles, then as a product of transpositions. Is σ even or odd?

> ***Solution.*** Following the cycles: $1 \to 3 \to 1$ and $2 \to 5 \to 2$ and $4 \to 6 \to 4$. Thus $\sigma = (1\ 3)(2\ 5)(4\ 6)$. As transpositions (already done!): $\sigma = (1\ 3)(2\ 5)(4\ 6)$. This is a product of 3 transpositions, so σ is odd, and $\operatorname{sgn}(\sigma) = -1$. ∎

7. In D_4, compute $(sr)(sr^2)$ using the presentation $D_4 = \langle r, s \mid r^4 = s^2 = e, sr = r^{-1}s \rangle$.

> ***Solution.*** $(sr)(sr^2) = s(rs)r^2 = s(r^{-1}s)r^2 = sr^{-1}sr^2 = sr^3sr^2$. Using $sr = r^{-1}s = r^3s$: $sr^3 = s(r^3) = (sr)r^2 = (r^3s)r^2 = r^3sr^2 = r^3(r^{-2}s) = r^3r^2s = rs$. Then $(rs)(r^2) = r^3$. Wait, let me recalculate more carefully: $(sr)(sr^2) = s(rs)r^2$. Using $rs = sr^{-1} = sr^3$: $s(sr^3)r^2 = s^2r^3r^2 = er^5 = r^5 = r^1 = r$. ∎

8. Show that the Klein four-group V_4 has exactly three subgroups of order 2.

> ***Solution.*** $V_4 = \{e, a, b, ab\}$ with $a^2 = b^2 = (ab)^2 = e$. The subgroups of order 2 are $\{e, a\}$, $\{e, b\}$, and $\{e, ab\}$. Each is cyclic of order 2. ∎

9. Use Cayley's theorem to represent V_4 as a subgroup of S_4.

> ***Solution.*** Using left multiplication on $V_4 = \{e, a, b, ab\}$: $\lambda_e = \text{id}$, λ_a swaps $e \leftrightarrow a$ and $b \leftrightarrow ab$, giving $(e\ a)(b\ ab)$. Similarly, $\lambda_b = (e\ b)(a\ ab)$ and $\lambda_{ab} = (e\ ab)(a\ b)$. This matches the double transposition subgroup of S_4 from Example (in 2.7.1). ∎

10. Determine the center $Z(D_5)$ of the dihedral group D_5.

> ***Solution.*** By Theorem 2.6.3(4), since 5 is odd, $Z(D_5) = \{e\}$. ∎

End of Chapter 2

Chapter 3: Subgroups and Cosets

Every group contains within it a hierarchy of smaller groups, each preserving the algebraic structure of the whole. These subgroups, and the way they partition the group into cosets, reveal the deep internal organization of algebraic systems.

3.1 Subgroups

> **Definition**
>
> A subset H of a group G is a **subgroup**, written $H \leq G$, if H is itself a group under the same operation as G.

> **Theorem**
>
> **(Subgroup Test)** A nonempty subset $H \subseteq G$ is a subgroup if and only if:
>
> 1. For all $a, b \in H$, we have $ab \in H$ (closure).
> 2. For all $a \in H$, we have $a^{-1} \in H$ (inverses).

> ***Proof.*** ($\Rightarrow$) If H is a subgroup, closure and inverses hold by definition. ($\Leftarrow$) Closure is given. Since $H \neq \emptyset$, pick $a \in H$. Then $a^{-1} \in H$ by (2), so $e = aa^{-1} \in H$ by (1). Associativity is inherited from G. ∎

> **Theorem**
>
> **(One-Step Subgroup Test)** A nonempty subset $H \subseteq G$ is a subgroup if and only if $ab^{-1} \in H$ for all $a, b \in H$.

> **Example**
>
> In $(\mathbb{Z}, +)$: $2\mathbb{Z} = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$ is a subgroup. Indeed, $n\mathbb{Z} \leq \mathbb{Z}$ for any n.

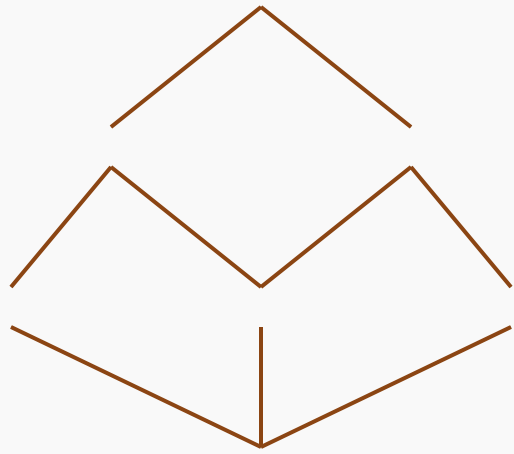
> **Example**
>
> In $GL_n(\mathbb{R})$: $SL_n(\mathbb{R}) = \{A : \det(A) = 1\}$ is a subgroup since $\det(AB) = \det(A)\det(B)$ and $\det(A^{-1}) = 1/\det(A)$.

3.2 Subgroup Lattices

The collection of all subgroups of G forms a lattice under inclusion, revealing the group's internal structure.

Example

Lattice of subgroups of $\mathbb{Z}_{12}$:



3.3 Cosets

Definition

Let $H \leq G$ and $a \in G$. The **left coset** of H by a is $aH = \{ah : h \in H\}$. The **right coset** is $Ha = \{ha : h \in H\}$.

Theorem

Let $H \leq G$. For $a, b \in G$, the following are equivalent:

1. $aH = bH$
2. $a \in bH$
3. $b^{-1}a \in H$

Proof. (1⇒2): $a = ae \in aH = bH$. (2⇒3): If $a \in bH$, then $a = bh$ for some $h \in H$, so $b^{-1}a = h \in H$. (3⇒1): If $b^{-1}a \in H$, say $b^{-1}a = h$, then $a = bh \in bH$, so $aH \subseteq bH$. Similarly $bH \subseteq aH$. ∎

Proposition

All left cosets of H in G have the same cardinality as H.

Proof. Define $\varphi : H \to aH$ by $\varphi(h) = ah$. This is bijective: if $ah_1 = ah_2$, then $h_1 = h_2$ by left cancellation (injective); for any $ah \in aH$, we have $\varphi(h) = ah$ (surjective). ∎

3.4 Lagrange's Theorem

Theorem

(Lagrange's Theorem) Let G be a finite group and $H \leq G$. Then $|H|$ divides $|G|$. Moreover, the number of distinct left cosets of H in G (called the **index** $[G : H]$) satisfies $|G| = |H| \cdot [G : H]$.

> ***Proof.*** The left cosets of H partition G: (i) every element is in some coset ($a \in aH$), (ii) distinct cosets are disjoint (if $aH \cap bH \neq \emptyset$, then $aH = bH$). Since each coset has $|H|$ elements and there are $[G : H]$ cosets, $|G| = |H| \cdot [G : H]$. ∎

If G is a finite group and $a \in G$, then $|a|$ divides $|G|$.

> ***Proof.*** $|\langle a \rangle| = |a|$ divides $|G|$ by Lagrange. ∎

If $|G| = p$ is prime, then G is cyclic.

> ***Proof.*** Let $a \in G$ with $a \neq e$. Then $|a| > 1$ and $|a|$ divides p, so $|a| = p$, thus $G = \langle a \rangle$. ∎

3.5 Applications

Theorem

(Fermat's Little Theorem) If p is prime and $\gcd(a, p) = 1$, then $a^{p-1} \equiv 1 \pmod{p}$.

> ***Proof.*** Consider $G = \mathbb{Z}_p^*$, which has order $p-1$. For $[a] \in G$, we have $|[a]|$ divides $p-1$ by Lagrange, so $[a]^{p-1} = [a^{p-1}] = [1]$, i.e., $a^{p-1} \equiv 1 \pmod{p}$. ∎

Theorem

(Euler's Theorem) If $\gcd(a, n) = 1$, then $a^{\varphi(n)} \equiv 1 \pmod{n}$.

> ***Proof.*** $|\mathbb{Z}_n^*| = \varphi(n)$, so $[a]^{\varphi(n)} = [1]$ by Lagrange. ∎

Greek Meditation: Hierarchy of Being

Plotinus envisioned reality as emanating from The One in descending levels: The One, Intellect, Soul, Matter—each level containing and being contained. So too with subgroups: within the whole group nest hierarchies of substructures, each a perfect group in miniature, each preserving the essence of the parent structure. The lattice of subgroups mirrors Plotinus' Great Chain of Being, a nested hierarchy where higher encompasses lower, yet each level possesses its own complete reality. To study subgroups is to trace the emanations of mathematical Being from unity to multiplicity.

Zen Meditation: Partitioning the Whole

A coset partition is like cutting a pie: each piece distinct, yet all equal in size, together reconstituting the whole without gap or

> overlap. There is profound unity in this division—the whole is not destroyed but revealed through its parts. *Form is emptiness, emptiness is form.* The group is its cosets; the cosets are the group. Neither exists independently. Lagrange's Theorem is not a constraint but a harmony: part and whole must agree, like the branches of a tree matching its trunk, like breaths matching heartbeats. Understanding this is understanding the interconnectedness of all things.

Practice Problems

1. Prove that the intersection of two subgroups is a subgroup.

> ***Solution.*** Let $H, K \leq G$. Since $e \in H$ and $e \in K$, we have $H \cap K \neq \emptyset$. If $a, b \in H \cap K$, then $ab^{-1} \in H$ (as H is a subgroup) and $ab^{-1} \in K$ (as K is a subgroup), so $ab^{-1} \in H \cap K$. By one-step test, $H \cap K \leq G$. ∎

2. List all left cosets of $\langle 3 \rangle$ in $\mathbb{Z}_{12}$.

> ***Solution.*** $H = \langle 3 \rangle = \{0, 3, 6, 9\}$. Cosets:
> $0 + H = \{0, 3, 6, 9\}$, $1 + H = \{1, 4, 7, 10\}$,
> $2 + H = \{2, 5, 8, 11\}$. These three cosets partition $\mathbb{Z}_{12}$. ∎

3. How many elements of order 7 does a group of order 28 have?

> ***Solution.*** If $|a| = 7$, then $\langle a \rangle$ has 7 elements, 6 of which (all except e) have order 7. Distinct cyclic subgroups of order 7 are disjoint except at identity. By Lagrange, there can be at

> most $28/7 = 4$ such subgroups. Each contributes 6 elements of order 7. However, existence is not guaranteed; it depends on group structure. If they all exist: $4 \times 6 = 24$ elements (but typically 0, 6, or some multiple of 6). ∎

4. Prove: if $|G| = pq$ where p, q are distinct primes, then any proper nontrivial subgroup is cyclic.

> ***Solution.*** Let $H < G$ with $H \neq \{e\}$ and $H \neq G$. By Lagrange, $|H|$ divides pq, so $|H| \in \{1, p, q, pq\}$. Since H is proper nontrivial, $|H| = p$ or $|H| = q$, both prime. Any group of prime order is cyclic. ∎

5. Verify Lagrange's Theorem for $H = \{e, (12)\} \leq S_3$.

> ***Solution.*** $|S_3| = 6$, $|H| = 2$. Left cosets:
> $eH = H = \{e, (12)\}$, $(13)H = \{(13), (132)\}$,
> $(23)H = \{(23), (123)\}$. Three cosets, each of size 2, covering all 6 elements. $[S_3 : H] = 3$ and
> $|H| \cdot [S_3 : H] = 2 \cdot 3 = 6 = |S_3|$. ✓ ∎

❦

End of Chapter 3

Chapter 4: Group Homomorphisms

Homomorphisms are the structure-preserving maps between groups, the morphisms that relate one algebraic universe to another. They allow us to compare groups, to understand one in terms of another, and to classify groups up to their essential structure.

4.1 Homomorphisms

> **Definition**
>
> A **homomorphism** from group $(G, \cdot)$ to $(H, *)$ is a function $\varphi : G \to H$ such that
>
> $$\varphi(a \cdot b) = \varphi(a) * \varphi(b)$$
>
> for all $a, b \in G$.

> **Proposition**
>
> Let $\varphi : G \to H$ be a homomorphism. Then:
>
> 1. $\varphi(e_G) = e_H$
> 2. $\varphi(a^{-1}) = \varphi(a)^{-1}$ for all $a \in G$
> 3. $\varphi(a^n) = \varphi(a)^n$ for all $a \in G$, $n \in \mathbb{Z}$

> ***Proof.*** (1) $\varphi(e_G) = \varphi(e_G e_G) = \varphi(e_G)\varphi(e_G)$, so $e_H = \varphi(e_G)$ by cancellation. (2) $e_H = \varphi(e_G) = \varphi(a a^{-1}) = \varphi(a)\varphi(a^{-1})$, so $\varphi(a^{-1}) = \varphi(a)^{-1}$. (3) Induction using $\varphi(a^{n+1}) = \varphi(a^n a) = \varphi(a^n)\varphi(a)$. ∎

> **Example**
>
> The **sign homomorphism** $\operatorname{sgn} : S_n \to \{\pm 1\}$ maps even permutations to 1 and odd to -1. This is a surjective homomorphism.

> **Example**
>
> The **determinant** $\det : GL_n(\mathbb{R}) \to \mathbb{R}^*$ is a homomorphism since $\det(AB) = \det(A)\det(B)$.

> **Example**
>
> For any $n \in \mathbb{Z}$, the map $\varphi : \mathbb{Z} \to \mathbb{Z}_n$ defined by $\varphi(a) = [a]$ is a surjective homomorphism.

4.2 Kernel and Image

> **Definition**
>
> Let $\varphi : G \to H$ be a homomorphism.
>
> - The **kernel** of φ is $\ker(\varphi) = \{g \in G : \varphi(g) = e_H\}$.
> - The **image** of φ is $\operatorname{im}(\varphi) = \{\varphi(g) : g \in G\} = \varphi(G)$.

Theorem

Let $\varphi : G \to H$ be a homomorphism. Then:

1. $\ker(\varphi) \leq G$
2. $\text{im}(\varphi) \leq H$
3. φ is injective if and only if $\ker(\varphi) = \{e_G\}$

Proof. (1) $e_G \in \ker(\varphi)$ since $\varphi(e_G) = e_H$. If $a, b \in \ker(\varphi)$, then $\varphi(ab^{-1}) = \varphi(a)\varphi(b^{-1}) = \varphi(a)\varphi(b)^{-1} = e_H e_H^{-1} = e_H$, so $ab^{-1} \in \ker(\varphi)$. (2) $e_H = \varphi(e_G) \in \text{im}(\varphi)$. If $\varphi(a), \varphi(b) \in \text{im}(\varphi)$, then $\varphi(a)\varphi(b)^{-1} = \varphi(a)\varphi(b^{-1}) = \varphi(ab^{-1}) \in \text{im}(\varphi)$. (3) ($\Rightarrow$) If φ is injective and $\varphi(a) = e_H = \varphi(e_G)$, then $a = e_G$. ($\Leftarrow$) If $\ker(\varphi) = \{e_G\}$ and $\varphi(a) = \varphi(b)$, then $\varphi(ab^{-1}) = \varphi(a)\varphi(b)^{-1} = e_H$, so $ab^{-1} \in \ker(\varphi) = \{e_G\}$, thus $ab^{-1} = e_G$, i.e., $a = b$. ∎

4.3 Isomorphisms

Definition

An **isomorphism** is a bijective homomorphism. Groups G and H are **isomorphic**, written $G \cong H$, if there exists an isomorphism $\varphi : G \to H$.

Isomorphic groups are algebraically identical—they differ only in the labels of their elements. All algebraic properties are preserved under isomorphism.

Theorem

If $\varphi : G \to H$ is an isomorphism, then $\varphi^{-1} : H \to G$ is also an isomorphism.

Theorem

(Cayley's Theorem) Every group G is isomorphic to a subgroup of some symmetric group S_X (where X is an appropriate set).

Proof. For each $g \in G$, define $\lambda_g : G \to G$ by $\lambda_g(x) = gx$ (left multiplication). This is a bijection with inverse $\lambda_{g^{-1}}$. Define $\varphi : G \to S_G$ (the group of permutations of G) by $\varphi(g) = \lambda_g$. This is a homomorphism: $\varphi(gh)(x) = \lambda_{gh}(x) = (gh)x = g(hx) = \lambda_g(\lambda_h(x)) = (\varphi(g) \circ \varphi(h))(x)$. If $\varphi(g) = \varphi(h)$, then $\lambda_g = \lambda_h$, so $ge = he$, thus $g = h$ (injective). Therefore $G \cong \mathrm{im}(\varphi) \leq S_G$. ∎

4.4 Automorphisms

Definition

An **automorphism** of G is an isomorphism $\varphi : G \to G$. The set of all automorphisms forms a group under composition, denoted $\mathrm{Aut}(G)$.

Definition

For $a \in G$, the **inner automorphism** induced by a is $\varphi_a : G \to G$ defined by $\varphi_a(x) = axa^{-1}$. The set of inner automorphisms forms a subgroup $\mathrm{Inn}(G) \leq \mathrm{Aut}(G)$.

Example

$\mathrm{Aut}(\mathbb{Z}) \cong \mathbb{Z}_2 = \{\mathrm{id}, -\mathrm{id}\}$ where $-\mathrm{id}$ sends n to $-n$.

Example

For cyclic group $\mathbb{Z}_n$, we have $\mathrm{Aut}(\mathbb{Z}_n) \cong \mathbb{Z}_n^*$, the group of units modulo n.

4.5 Isomorphism Theorems

Theorem

(First Isomorphism Theorem) Let $\varphi : G \to H$ be a homomorphism. Then $G/\ker(\varphi) \cong \mathrm{im}(\varphi)$.

> ***Proof.*** Define $\overline{\varphi} : G/\ker(\varphi) \to \mathrm{im}(\varphi)$ by $\overline{\varphi}(a\ker(\varphi)) = \varphi(a)$. Well-defined: if $a\ker(\varphi) = b\ker(\varphi)$, then $b^{-1}a \in \ker(\varphi)$, so $\varphi(b^{-1}a) = e_H$, thus $\varphi(a) = \varphi(b)$. Homomorphism: $\overline{\varphi}(aK \cdot bK) = \overline{\varphi}(abK) = \varphi(ab) = \varphi(a)\varphi(b) = \overline{\varphi}(aK)\overline{\varphi}(bK)$. Injective: if $\overline{\varphi}(aK) = e_H$, then $\varphi(a) = e_H$, so $a \in \ker(\varphi) = K$, thus $aK = K$. Surjective by definition of image. ∎

Theorem

(Second Isomorphism Theorem) Let $H \leq G$ and $N \trianglelefteq G$. Then $H/(H \cap N) \cong HN/N$.

Theorem

(Third Isomorphism Theorem) Let $K \trianglelefteq H \trianglelefteq G$. Then $(G/K)/(H/K) \cong G/H$.

Greek Meditation: Morphe and Eidos

> Aristotle distinguished *morphe* (form) from *hyle* (matter). The form of a statue—its shape, proportions, symmetries—can be instantiated in bronze, marble, or clay. The material changes, but the form persists. An isomorphism is precisely this: two groups sharing the same form (*eidos*) despite different material realizations. When we say $\mathbb{Z}_4 \cong \langle i \rangle$ (in $\mathbb{C}^*$), we recognize the same four-fold cyclic structure in modular arithmetic and complex multiplication. Mathematics studies forms, not matter—and homomorphisms are the maps that reveal when different structures share essential form.

> **Zen Meditation: The Map and the Territory**
>
> A homomorphism is like a map of a landscape: it preserves relationships (this city north of that, this road connecting these towns) while potentially omitting details (exact distances, elevations). The kernel is what the map ignores—all points mapped to a single point. An isomorphism is a perfect map, preserving everything essential, translating fully between territories. Yet even in this perfection, remember: *the map is not the territory*. $\mathbb{Z}$ and $2\mathbb{Z}$ are isomorphic, identical in structure, yet different in realization. Structure and substance dance together; to grasp one without the other is to miss the whole.

Practice Problems

1. Prove that if $\varphi : G \to H$ is a homomorphism and $K \leq G$, then $\varphi(K) \leq H$.

> ***Solution.*** Since $e_G \in K$, we have $e_H = \varphi(e_G) \in \varphi(K)$, so $\varphi(K) \neq \emptyset$. If $\varphi(a), \varphi(b) \in \varphi(K)$ (with $a, b \in K$), then $\varphi(a)\varphi(b)^{-1} = \varphi(a)\varphi(b^{-1}) = \varphi(ab^{-1}) \in \varphi(K)$ since $ab^{-1} \in K$. ∎

2. Find ker and im of $\det : GL_3(\mathbb{R}) \to \mathbb{R}^*$.

> ***Solution.*** $\ker(\det) = \{A \in GL_3(\mathbb{R}) : \det(A) = 1\} = SL_3(\mathbb{R})$. $\operatorname{im}(\det) = \mathbb{R}^*$ since for any $r \neq 0$, the matrix $\operatorname{diag}(r, 1, 1)$ has determinant r. ∎

3. Show that $\mathbb{Z}_6 \ncong S_3$ despite both having order 6.

> ***Solution.*** $\mathbb{Z}_6$ is abelian while S_3 is not (e.g., $(12)(13) \neq (13)(12)$). Since isomorphisms preserve commutativity, $\mathbb{Z}_6 \ncong S_3$. ∎

4. Prove: $\operatorname{Aut}(\mathbb{Z}_n) \cong \mathbb{Z}_n^*$.

> ***Solution.*** Any $\varphi \in \operatorname{Aut}(\mathbb{Z}_n)$ is determined by $\varphi([1])$ since $\mathbb{Z}_n = \langle [1] \rangle$. We need $\langle \varphi([1]) \rangle = \mathbb{Z}_n$, so $\varphi([1])$ must be a generator, i.e., $\varphi([1]) = [a]$ with $\gcd(a, n) = 1$. Define $\Phi : \operatorname{Aut}(\mathbb{Z}_n) \to \mathbb{Z}_n^*$ by $\Phi(\varphi) = \varphi([1])$. This is a bijection and preserves multiplication: $(\varphi \circ \psi)([1]) = \varphi(\psi([1])) = \varphi([b]) = [ab]$. ∎

5. Use First Isomorphism Theorem to show $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$.

> ***Solution.*** Define $\varphi : \mathbb{Z} \to \mathbb{Z}_n$ by $\varphi(a) = [a]$. This is a surjective homomorphism with $\ker(\varphi) = n\mathbb{Z}$. By FIT, $\mathbb{Z}/\ker(\varphi) \cong \operatorname{im}(\varphi)$, i.e., $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$. ∎

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End of Chapter 4

Chapter 5: Quotient Groups and Normal Subgroups

In quotient groups, we see algebra's power to collapse distinctions, treating entire cosets as single elements. This construction, fundamental to understanding group structure, requires the special class of normal subgroups—those subgroups invariant under conjugation.

5.1 Normal Subgroups

> **Definition**
>
> A subgroup N of G is **normal**, written $N \trianglelefteq G$, if $gN = Ng$ for all $g \in G$. Equivalently, $gNg^{-1} = N$ for all $g \in G$.

Proposition

The following are equivalent for $N \leq G$:

1. $N \trianglelefteq G$
2. $gNg^{-1} \subseteq N$ for all $g \in G$
3. $gng^{-1} \in N$ for all $g \in G$ and $n \in N$

Theorem

(Normal Subgroup Test) Let $N \leq G$. Then $N \trianglelefteq G$ if and only if N is the kernel of some homomorphism from G.

Example

Every subgroup of an abelian group is normal (since $gng^{-1} = gg^{-1}n = n$).

Example

In S_3, the subgroup $H = \{e, (12)\}$ is NOT normal: $(13)(12)(13)^{-1} = (13)(12)(13) = (23) \notin H$.

Example

$A_n \trianglelefteq S_n$ for all n since $A_n = \ker(\text{sgn})$.

Theorem

If $[G : N] = 2$, then $N \trianglelefteq G$.

Proof. For any $g \in G$, either $g \in N$ or $g \notin N$. If $g \in N$, then $gN = N = Ng$. If $g \notin N$, the two cosets of N are N and gN (left cosets), and also N and Ng (right cosets). Since $g \in gN$ and $g \in Ng$, and both sets partition G into two cosets one of which is N, we must have $gN = Ng$. ∎

5.2 Quotient Groups

Definition

Let $N \trianglelefteq G$. The **quotient group** G/N is the set of left cosets $\{gN : g \in G\}$ with operation $(aN)(bN) = (ab)N$.

Theorem

If $N \trianglelefteq G$, then G/N is a group under the operation $(aN)(bN) = (ab)N$. Moreover, $|G/N| = [G : N]$.

> ***Proof.*** Well-defined: Suppose $aN = a'N$ and $bN = b'N$. Then $a' = an_1$ and $b' = bn_2$ for some $n_1, n_2 \in N$. Thus $a'b' = an_1bn_2 = a(n_1bn_2) = ab(b^{-1}n_1bn_2)$. Since N is normal, $b^{-1}n_1b \in N$, so $b^{-1}n_1bn_2 \in N$, thus $a'b'N = abN$. Identity: $eN = N$ satisfies $(eN)(gN) = (eg)N = gN$. Inverses: $(gN)^{-1} = g^{-1}N$ since $(gN)(g^{-1}N) = (gg^{-1})N = eN = N$. Associativity: inherited from G. ∎

> **Example**
>
> $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$ via $a + n\mathbb{Z} \mapsto [a]$.

> **Example**
>
> $GL_n(\mathbb{R})/SL_n(\mathbb{R})$ has structure isomorphic to $(\mathbb{R}^*, \cdot)$ via the determinant map.

> **Example**
>
> $S_n/A_n \cong \mathbb{Z}_2$ via the sign homomorphism.

5.3 Simple Groups

Definition

A nontrivial group G is **simple** if its only normal subgroups are $\{e\}$ and G itself.

Simple groups are the "atoms" of group theory—they cannot be decomposed further via quotients.

Theorem

A group G of prime order p is simple.

> ***Proof.*** If $N \trianglelefteq G$ with $N \neq \{e\}$, then $|N|$ divides $|G| = p$ by Lagrange. Since p is prime and $|N| > 1$, we have $|N| = p$, so $N = G$. ∎

Theorem

A_n is simple for $n \geq 5$.

(Proof omitted—requires detailed analysis of conjugacy classes in A_n.)

Example

A_5 (order 60) is the smallest nonabelian simple group.

5.4 Composition Series

> **Definition**
>
> A **composition series** for G is a sequence of subgroups
>
> $$\{e\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$$
>
> where each quotient G_{i+1}/G_i is simple. The simple quotients are called **composition factors**.

> **Theorem**
>
> **(Jordan-Hölder Theorem)** Any two composition series for a finite group G have the same length, and the composition factors are the same up to isomorphism and reordering.

This profound result shows that while groups can be "factored" in different ways, the "prime factors" (simple quotients) are essentially unique—an analogue of the Fundamental Theorem of Arithmetic for groups.

Example

For $\mathbb{Z}_{12}$, one composition series is:

$$\{0\} \trianglelefteq \langle 6 \rangle \trianglelefteq \langle 2 \rangle \trianglelefteq \mathbb{Z}_{12}$$

with factors $\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_3$ (in some order). Another is:

$$\{0\} \trianglelefteq \langle 4 \rangle \trianglelefteq \langle 2 \rangle \trianglelefteq \mathbb{Z}_{12}$$

with factors $\mathbb{Z}_3, \mathbb{Z}_2, \mathbb{Z}_2$. Same factors, different order.

5.5 Applications to Galois Theory

The structure of quotient groups and normal subgroups underlies Galois theory's solution to the problem of solving polynomial equations by radicals.

Definition

A group G is **solvable** if it has a composition series with all abelian composition factors.

Theorem

(Galois, informal) A polynomial equation is solvable by radicals if and only if its Galois group is solvable.

Since A_5 is simple and nonabelian, any polynomial with Galois group A_5 cannot be solved by radicals—giving the classic result that there is no general formula for quintic equations.

> **Greek Meditation: The One and the Many**
>
> Heraclitus proclaimed *panta rhei*—all things flow, unity emerging from strife of opposites. Parmenides countered: only The One truly exists, changeless and eternal. In quotient groups, this tension resolves dialectically. We take the many (elements of G) and collapse them into the one (a single coset). Yet the coset itself is many elements unified. G/N is simultaneously simpler (fewer "elements"—the cosets) and reveals hidden structure. We ascend from multiplicity to unity while preserving essential relationships. The quotient is both less and more than the original: less in detail, more in clarity. As Plotinus taught: each level of reality both emanates from and returns to The One.

> **Zen Meditation: Collapsing Distinctions**
>
> In Zen, we learn that distinctions—self/other, subject/object, nirvana/samsara—are ultimately conventional, not absolute. The quotient group embodies this wisdom. In forming G/N, we declare: all elements of a coset are *the same* for our purposes. We collapse distinctions, seeing equivalence where before we saw difference. Yet this is not nihilism or confusion—we preserve exactly what matters (the group structure) while releasing what doesn't (distinctions within cosets). Like a Zen

> master pointing beyond words to the moon, the quotient map $\pi : G \to G/N$ points beyond elements to structure. *Two hands clap and make a sound; what is the sound of one coset?*

Practice Problems

1. Prove that if $N \trianglelefteq G$ and $M \trianglelefteq G$, then $N \cap M \trianglelefteq G$.

> ***Solution.*** We know $N \cap M \leq G$. For normality: let $g \in G$ and $x \in N \cap M$. Then $x \in N$ and $x \in M$. Since $N \trianglelefteq G$, we have $gxg^{-1} \in N$. Since $M \trianglelefteq G$, we have $gxg^{-1} \in M$. Thus $gxg^{-1} \in N \cap M$. ∎

2. Find all normal subgroups of S_3.

> ***Solution.*** By checking: $\{e\}$ and S_3 are always normal. $A_3 = \{e, (123), (132)\}$ has index 2, so $A_3 \trianglelefteq S_3$. Subgroups of order 2 (like $\{e, (12)\}$) are NOT normal (conjugation moves transpositions around). Answer: $\{e\}$, A_3, S_3. ∎

3. Describe the quotient group $\mathbb{Z}_{24}/\langle 6 \rangle$.

> ***Solution.*** $H = \langle 6 \rangle = \{0, 6, 12, 18\}$ has order 4, so $|\mathbb{Z}_{24}/H| = 24/4 = 6$. The quotient is cyclic of order 6, so $\mathbb{Z}_{24}/\langle 6 \rangle \cong \mathbb{Z}_6$. ∎

4. Show that $\mathbb{Z}_p$ is simple for prime p.

> ***Solution.*** If $N \trianglelefteq \mathbb{Z}_p$ with $N \neq \{0\}$, then $|N|$ divides p by Lagrange. Since p is prime and $|N| > 1$, we have $|N| = p$, so $N = \mathbb{Z}_p$. Thus only normal subgroups are $\{0\}$ and $\mathbb{Z}_p$. ∎

5. Find a composition series for $\mathbb{Z}_{30}$.

> ***Solution.*** $\mathbb{Z}_{30} = \langle 1 \rangle$ has order $30 = 2 \cdot 3 \cdot 5$. One series:
>
> $$\{0\} \trianglelefteq \langle 15 \rangle \trianglelefteq \langle 5 \rangle \trianglelefteq \langle 1 \rangle = \mathbb{Z}_{30}$$
>
> Orders: 1, 2, 6, 30. Quotient orders: 2, 3, 5 (all prime, hence simple). Factors: $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_5$. ∎

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End of Chapter 5

∘ End of Volume I ∘

Volume 2: The Instant Between States

Πάντα ῥεῖ καὶ οὐδὲν μένει

Everything flows—

yet catch the instant of transformation,

that liminal threshold where one state becomes another,

and you have grasped eternity.

A square rotates ninety degrees. For an immeasurable instant—neither in the old position nor fully in the new—it exists in pure transition. This instant has no duration, occupies no time, yet contains the essence of transformation itself. The group operation is this instant made mathematical: not the before, not the after, but the *how* of change.

Heraclitus watched rivers flowing and declared permanence impossible—we cannot step twice into the same water. Yet he missed the deeper truth: the river's form persists precisely *because* the water flows. Pattern endures through transformation. Identity survives change. This is the paradox group theory resolves: how can everything alter while something remains? The answer is symmetry—not as static balance, but as dynamic invariance, structure maintained through motion.

On Transformation and Time:

Here is a question physics cannot answer but mathematics illuminates: what is the difference between time and transformation?

We measure time by change—the sun's motion, the pendulum's swing, the cesium atom's oscillation. Remove all change and time becomes meaningless, undetectable, perhaps nonexistent. What if time itself *is* transformation? What if the forward flow we call duration is actually the successive application of symmetry operations on the state of the universe?

A group action maps a set to itself through transformation. Time maps the universe to itself through successive states. The formal parallel is striking: both are operations that combine associatively, both possess an identity (the null transformation, the present moment), both can be inverted (at least formally). Could the physical arrow of time be related to the algebraic structure of transformation groups? This volume does not answer such questions—it teaches you the mathematical language in which they might one day be posed precisely.

On Invariance and Essence:

The Sylow theorems reveal how large groups decompose into smaller pieces, how complexity arises from simplicity recombining. But they tell us something stranger: structure determines its own internal architecture. Given a group's order, we can predict constraints on its subgroups, limitations on its form. The group doesn't just possess structure—it *is* structure manifesting as mathematical object.

When we study group actions, we observe transformation in its purest form. A group acts upon a set: each element becomes an operator, each operation a transformation. The orbit of a point is the set of all positions it can reach through symmetry. The stabilizer is the symmetry that leaves it unchanged. In this interplay between

motion and stillness, between orbit and stabilizer, we see the fundamental dialectic of mathematics: change and preservation, flow and form, becoming and being.

On Rings and Multiplication's Entry:

Addition generates groups—the simplest, most natural operation. But the world requires multiplication as well, that second operation where elements don't just combine but interact, where structure becomes richer, stranger. Rings are spaces where two operations coexist: addition preserving its group structure, multiplication obeying different laws, the two intertwined through the distributive property.

In rings, we encounter ideals—subsets that absorb multiplication from the outside while maintaining their integrity. These are boundaries within algebraic space, membranes that allow certain operations while blocking others. Through quotient rings, we learn how structure can collapse onto simpler forms without losing essential information—how complexity can be folded into simplicity, how the ten thousand things can return to the one.

The Question That Remains:

Every transformation implies something unchanged—else how would we recognize that transformation occurred? To say "X became Y" requires that some underlying identity persists, witnessing both states. This observer, this invariant core, is what symmetry formalizes. But here's the puzzle: when consciousness observes its own transformation—when you remember being younger, different, other than you are now—what remains unchanged to witness the

change? What is *your* invariant? Which transformation group describes the operations that leave your identity intact?

These are not idle questions. Mathematics cannot answer them, but it can teach precision in asking.

> **Greek Wisdom — Ἑλληνική Σοφία**
>
> *"ἁρμονία ἀφανὴς φανερῆς κρείττων" — Hidden harmony is superior to visible harmony*
>
> Heraclitus understood that surface appearance deceals deeper pattern. The symmetries we study here are hidden in plain sight —rotations that leave objects seemingly unchanged, permutations that rearrange while preserving structure. The most profound symmetries are those least obvious: the automorphisms of algebraic systems, the invariances under abstract operations. To perceive hidden harmony is to see past the superficial into the structural. Group theory is the science of hidden harmonies made explicit.

> **Zen Reflection — 禅の省察**
>
> *The bamboo casts its shadow on the stair, but the shadow does not move. The moon travels across the sky, reflected in the pond, yet the pond's moon is still.*
>
> Transformation and invariance are not opposites—they define each other. The bamboo's shadow moves because the sun moves, yet we perceive the shadow as static. The moon travels,

> yet its reflection seems eternal. Group actions reveal this same interplay: movement that creates stillness, change that enables permanence. The Japanese concept of 無常 (mujō)—impermanence—is not nihilistic. It recognizes that only through change does pattern become visible, that only through transformation does structure reveal itself. In the instant between states, eternity lives.

Step into the threshold. Learn to see transformation not as disruption but as revelation, change not as loss but as the method by which form makes itself known. Between one state and the next, in that immeasurable instant, mathematical truth awaits.

❦

Chapter 6: Group Actions

Group actions provide a unifying framework for understanding how groups operate on sets, transforming abstract algebraic structures into concrete geometric and combinatorial phenomena. This powerful concept illuminates permutation groups, conjugacy, and counting problems through a single elegant lens.

6.1 Definition and Examples

> **Definition**
>
> A **left group action** of a group G on a set X is a function $\cdot : G \times X \to X$, written $(g, x) \mapsto g \cdot x$, satisfying:
>
> 1. **(Identity)** $e \cdot x = x$ for all $x \in X$
> 2. **(Compatibility)** $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G$ and $x \in X$
>
> When the action is clear from context, we write gx instead of $g \cdot x$.

A **right action** satisfies $x \cdot e = x$ and $x \cdot (gh) = (x \cdot g) \cdot h$. Unless specified otherwise, "action" means left action.

> **Proposition**
>
> A group action $G \times X \to X$ is equivalent to a homomorphism $\varphi : G \to \mathrm{Sym}(X)$, where $\mathrm{Sym}(X)$ is the group of all permutations of X.

Proof. Given an action, define $\varphi(g) : X \to X$ by $\varphi(g)(x) = g \cdot x$. Then $\varphi(g)$ is bijective with inverse $\varphi(g^{-1})$, and $\varphi(gh)(x) = (gh) \cdot x = g \cdot (h \cdot x) = \varphi(g)(\varphi(h)(x))$, so $\varphi(gh) = \varphi(g) \circ \varphi(h)$. Conversely, given $\varphi : G \to \mathrm{Sym}(X)$, define $g \cdot x = \varphi(g)(x)$. ∎

Example

(Conjugation) G acts on itself by conjugation: $g \cdot x = gxg^{-1}$. Identity: $e \cdot x = exe^{-1} = x$. Compatibility: $(gh) \cdot x = (gh)x(gh)^{-1} = g(hxh^{-1})g^{-1} = g \cdot (h \cdot x)$.

Example

(Left Multiplication) G acts on itself by left multiplication: $g \cdot x = gx$. This is always a faithful action (no non-identity element acts trivially).

Example

(On Cosets) Let $H \leq G$. Then G acts on the set of left cosets $G/H = \{gH : g \in G\}$ by $g \cdot (aH) = (ga)H$. This is well-defined since if $aH = bH$, then $b = ah$ for some $h \in H$, so $(ga)H = (gah)H = (gb)H$.

Example

(Symmetric Group) S_n acts on $\{1, 2, \ldots, n\}$ by $\sigma \cdot i = \sigma(i)$. More generally, S_n acts on any n-element set.

Example

(Dihedral Group) D_n acts on the vertices of a regular n-gon by rotations and reflections, realizing D_n as a subgroup of S_n.

Definition

An action is **faithful** (or **effective**) if different group elements induce different permutations, i.e., if $g \cdot x = x$ for all $x \in X$ implies $g = e$. Equivalently, the corresponding homomorphism $\varphi : G \to \mathrm{Sym}(X)$ is injective.

Definition

An action is **transitive** if for any $x, y \in X$, there exists $g \in G$ with $g \cdot x = y$. That is, X is a single orbit.

Example

G acting on G/H by left multiplication is transitive: given $aH, bH \in G/H$, the element ba^{-1} satisfies $(ba^{-1}) \cdot (aH) = bH$.

6.2 Orbits and Stabilizers

Definition

Let G act on X, and let $x \in X$. The **orbit** of x is

$$\mathrm{Orb}(x) = G \cdot x = \{g \cdot x : g \in G\}.$$

The **stabilizer** of x is

$$\mathrm{Stab}(x) = G_x = \{g \in G : g \cdot x = x\}.$$

Proposition

$\mathrm{Stab}(x)$ is a subgroup of G for any $x \in X$.

Proof. $e \cdot x = x$, so $e \in \mathrm{Stab}(x)$. If $g, h \in \mathrm{Stab}(x)$, then $(gh) \cdot x = g \cdot (h \cdot x) = g \cdot x = x$, so $gh \in \mathrm{Stab}(x)$. If $g \in \mathrm{Stab}(x)$, then $x = e \cdot x = (g^{-1}g) \cdot x = g^{-1} \cdot (g \cdot x) = g^{-1} \cdot x$, so $g^{-1} \in \mathrm{Stab}(x)$. ∎

Proposition

The orbits of a group action partition X.

Proof. Define $x \sim y$ if $y \in \mathrm{Orb}(x)$, i.e., if $y = g \cdot x$ for some $g \in G$. This is an equivalence relation: (Reflexive) $x = e \cdot x \in \mathrm{Orb}(x)$. (Symmetric) If $y = g \cdot x$, then $x = g^{-1} \cdot y$. (Transitive) If $y = g \cdot x$ and $z = h \cdot y$, then $z = h \cdot (g \cdot x) = (hg) \cdot x$. The orbits are the equivalence classes. ∎

Theorem

(Orbit-Stabilizer Theorem) Let G act on X and let $x \in X$. Then

$$|\mathrm{Orb}(x)| = [G : \mathrm{Stab}(x)] = \frac{|G|}{|\mathrm{Stab}(x)|}.$$

Proof. Define $\phi : G/\mathrm{Stab}(x) \to \mathrm{Orb}(x)$ by $\phi(g \cdot \mathrm{Stab}(x)) = g \cdot x$.

Well-defined: If $g\mathrm{Stab}(x) = h\mathrm{Stab}(x)$, then $h^{-1}g \in \mathrm{Stab}(x)$, so $(h^{-1}g) \cdot x = x$. Thus $g \cdot x = h \cdot ((h^{-1}g) \cdot x) = h \cdot x$.

Injective: If $g \cdot x = h \cdot x$, then $(h^{-1}g) \cdot x = h^{-1} \cdot (g \cdot x) = h^{-1} \cdot (h \cdot x) = x$, so $h^{-1}g \in \mathrm{Stab}(x)$, thus $g\mathrm{Stab}(x) = h\mathrm{Stab}(x)$.

Surjective: For any $y \in \mathrm{Orb}(x)$, we have $y = g \cdot x$ for some g, so $\phi(g\mathrm{Stab}(x)) = y$.

Therefore $|\mathrm{Orb}(x)| = |G/\mathrm{Stab}(x)| = [G : \mathrm{Stab}(x)]$. ∎

If G is finite and acts on X, then for any $x \in X$, we have $|\mathrm{Orb}(x)|$ divides $|G|$.

> **Example**
>
> Consider D_4 acting on the vertices of a square labeled $\{1,2,3,4\}$. The orbit of vertex 1 is $\{1,2,3,4\}$ (the action is transitive). The stabilizer of vertex 1 consists of the identity and the reflection through the axis passing through vertex 1, so $|\mathrm{Stab}(1)| = 2$. Indeed, $|\mathrm{Orb}(1)| = 4 = 8/2 = |D_4|/|\mathrm{Stab}(1)|$.

6.3 Cayley's Theorem Revisited

> **Theorem**
>
> **(Cayley's Theorem)** Every group G is isomorphic to a subgroup of $\mathrm{Sym}(G)$.

> ***Proof.*** G acts on itself by left multiplication: $g \cdot x = gx$. This induces a homomorphism $\varphi : G \to \mathrm{Sym}(G)$ with $\varphi(g)(x) = gx$. This action is faithful: if $g \cdot x = x$ for all $x \in G$, then in particular $g \cdot e = e$, so $g = e$. Thus $\ker(\varphi) = \{e\}$, so φ is injective. Therefore $G \cong \mathrm{Im}(\varphi) \leq \mathrm{Sym}(G)$. ∎

Cayley's Theorem shows every abstract group is isomorphic to a concrete group of permutations. For finite groups, this means $G \cong$ a subgroup of S_n where $n = |G|$.

> **Example**
>
> The cyclic group $C_4 = \langle r \rangle$ with $r^4 = e$ embeds into S_4 via left multiplication:
>
> $$\varphi(r) = (1\ 2\ 3\ 4), \quad \varphi(r^2) = (1\ 3)(2\ 4), \quad \varphi(r^3) = (1\ 4\ 3\ 2).$$

Definition

The **regular representation** of G is the homomorphism $G \to \mathrm{Sym}(G)$ given by left multiplication.

6.4 Conjugacy Classes

Definition

Elements $x, y \in G$ are **conjugate** if $y = gxg^{-1}$ for some $g \in G$. The **conjugacy class** of x is

$$\mathrm{Cl}(x) = \{gxg^{-1} : g \in G\}.$$

Conjugacy is an equivalence relation, and the conjugacy classes partition G. Note that $\mathrm{Cl}(x)$ is the orbit of x under the conjugation action of G on itself.

Definition

The **centralizer** of $x \in G$ is

$$C_G(x) = \{g \in G : gx = xg\} = \{g \in G : gxg^{-1} = x\}.$$

The **center** of G is

$$Z(G) = \{z \in G : zg = gz \text{ for all } g \in G\} = \bigcap_{x \in G} C_G(x).$$

Proposition

$C_G(x) = \mathrm{Stab}(x)$ for the conjugation action.

> ***Proof.*** By definition, $g \in \mathrm{Stab}(x)$ if and only if $g \cdot x = x$, i.e., $gxg^{-1} = x$, which is equivalent to $gx = xg$, i.e., $g \in C_G(x)$. ∎

Theorem

(Class Equation) Let G be a finite group. Then

$$|G| = |Z(G)| + \sum_i [G : C_G(x_i)],$$

where the sum is over representatives x_i of the non-central conjugacy classes (classes with more than one element).

Proof. The conjugacy classes partition G. An element $x \in Z(G)$ if and only if $\mathrm{Cl}(x) = \{x\}$ (singleton conjugacy class). For $x \notin Z(G)$, by the Orbit-Stabilizer Theorem,

$$|\mathrm{Cl}(x)| = |\mathrm{Orb}(x)| = [G : \mathrm{Stab}(x)] = [G : C_G(x)].$$

Summing over all classes:

$$|G| = \sum_{x \in Z(G)} |\mathrm{Cl}(x)| + \sum_{x_i \notin Z(G)} |\mathrm{Cl}(x_i)| = |Z(G)| + \sum_i [G : C_G(x_i)].$$

∎

Example

For S_3: The center is $\{e\}$. Conjugacy classes: $\{e\}$, $\{(12), (13), (23)\}$ (transpositions), $\{(123), (132)\}$ (3-cycles). Thus $6 = 1 + 3 + 2$.

Definition

The **normalizer** of a subgroup $H \leq G$ is

$$N_G(H) = \{g \in G : gHg^{-1} = H\}.$$

Proposition

$N_G(H)$ is the largest subgroup of G in which H is normal.

6.5 Applications

Application 1: Centers of p-Groups

> **Theorem**
>
> If G is a group of order p^n where p is prime and $n \geq 1$, then $Z(G) \neq \{e\}$.

> ***Proof.*** By the Class Equation, $|G| = |Z(G)| + \sum_i [G : C_G(x_i)]$ where each $[G : C_G(x_i)] > 1$. Since $x_i \notin Z(G)$, we have $C_G(x_i) \subsetneq G$, so $|C_G(x_i)| < |G| = p^n$. Thus $[G : C_G(x_i)] = |G|/|C_G(x_i)|$ is divisible by p (since both numerator and denominator are powers of p, and the denominator is a proper divisor). Therefore p divides each term $[G : C_G(x_i)]$. Since p divides $|G|$ and each $[G : C_G(x_i)]$, we have p divides $|Z(G)|$. Thus $|Z(G)| \geq p > 1$. ∎

Application 2: Burnside's Lemma

> **Theorem**
>
> **(Burnside's Lemma)** Let G act on a finite set X. The number of orbits is
>
> $$\frac{1}{|G|} \sum_{g \in G} |X^g|,$$
>
> where $X^g = \{x \in X : g \cdot x = x\}$ is the set of fixed points of g.

> ***Proof.*** Consider the set $S = \{(g, x) : g \cdot x = x\}$. Count $|S|$ in two ways. First, $|S| = \sum_{g \in G} |X^g|$. Second, $|S| = \sum_{x \in X} |\mathrm{Stab}(x)|$. By Orbit-Stabilizer, $|\mathrm{Stab}(x)| = |G|/|\mathrm{Orb}(x)|$. If X has k orbits $O_1, \ldots, O_k$, then
>
> $$|S| = \sum_{x \in X} \frac{|G|}{|\mathrm{Orb}(x)|} = \sum_{i=1}^{k} \sum_{x \in O_i} \frac{|G|}{|O_i|} = \sum_{i=1}^{k} |O_i| \cdot \frac{|G|}{|O_i|} = k|G|.$$
>
> Therefore $k = |S|/|G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$. ∎

Example

(Coloring a Square) How many ways can we color the vertices of a square with 2 colors (say red and blue), up to symmetries? D_4 acts on the set of $2^4 = 16$ colorings. Count fixed points for each symmetry:

- Identity e: all 16 colorings fixed
- Rotation r by 90°: only monochromatic colorings fixed (2 colorings)
- Rotation r^2 by 180°: opposite vertices same color (4 colorings)
- Rotation r^3 by 270°: only monochromatic (2 colorings)
- Reflections (4 of them): depends on axis, typically 4 colorings each

By Burnside: number of distinct colorings
$= \frac{1}{8}(16 + 2 + 4 + 2 + 4 + 4 + 4 + 4) = \frac{40}{8} = 5.$

Greek Meditation: Action and Being—The Dance of Transformation

Aristotle distinguished *energeia* (actuality, being-at-work) from *dynamis* (potentiality, capacity). A group action embodies this distinction: the group G possesses *dynamis*—the potential to transform the set X—while the action itself is *energeia*, the actualization of symmetry. Each group element is simultaneously actor and act, agent and agency. The orbit reveals what is possible through transformation; the stabilizer reveals what remains unchanged, echoing Parmenides' eternal unchanging Being amidst Heraclitean flux. In group actions, we witness the eternal forms of transformation—patterns that transcend particular instances yet manifest in concrete symmetries. The mathematician, like Plato's philosopher emerging from the cave, ascends from particular symmetries to their universal essence.

Zen Meditation: The Actor and the Acted Upon—Non-Duality in Group Actions

In the conjugation action $g \cdot x = gxg^{-1}$, the group acts upon itself. Subject and object collapse—the actor is the acted upon. This mirrors the Zen teaching of non-duality: the observer and observed are one. The stabilizer

> asks: *What remains when all has moved?* It points to stillness within motion, the unborn within birth-and-death. The orbit asks: *How far can transformation reach?* It reveals interconnection—seemingly distinct elements united by group action. Burnside's Lemma teaches: to count distinct forms, average over all perspectives. No single viewpoint captures reality; only by encompassing all symmetries do we see truly. Like the mirror that reflects without attachment, the group acts without grasping. *What is the orbit of your original face before you were born?*

Practice Problems

1. Verify that conjugation $g \cdot x = gxg^{-1}$ defines a group action of G on itself.

> ***Solution.*** Identity: $e \cdot x = exe^{-1} = x$. Compatibility: $(gh) \cdot x = (gh)x(gh)^{-1} = (gh)x(h^{-1}g^{-1}) = g(hxh^{-1})g^{-1} = g \cdot (h \cdot x)$. ∎

2. Show that the action of G on G/H by left multiplication is transitive.

> ***Solution.*** Given $aH, bH \in G/H$, we need $g \in G$ with $g \cdot (aH) = bH$. Take $g = ba^{-1}$. Then $g \cdot (aH) = (ba^{-1})(aH) = bH$. ∎

3. Find the orbits and stabilizers for the action of D_3 on the vertices of an equilateral triangle.

> ***Solution.*** Label vertices $\{1, 2, 3\}$. The action is transitive: $\mathrm{Orb}(1) = \{1, 2, 3\}$. For $\mathrm{Stab}(1)$: the identity and the reflection fixing vertex 1, so $|\mathrm{Stab}(1)| = 2$. By Orbit-Stabilizer: $|\mathrm{Orb}(1)| = 3 = 6/2 = |D_3|/|\mathrm{Stab}(1)|$. ∎

4. Prove that if G acts transitively on X and $|X| > 1$, then there exists $g \in G$ with $g \neq e$ such that g has no fixed points.

> ***Solution.*** Suppose every non-identity element has a fixed point. Pick distinct $x, y \in X$. By transitivity, there exists g with $g \cdot x = y$. Since

> $x \neq y$, we have $g \neq e$. By assumption, g fixes some $z \in X$. If every non-identity has a fixed point and we have only finitely many elements acting on finitely many points, we reach a contradiction with counting arguments (details require finite case analysis). For infinite cases or general proof, use more sophisticated techniques. A direct approach: if $|X| = n > 1$ and $|G|$ is finite, count (g, x) pairs where $g \cdot x = x$. Total is $\sum_g |X^g| = \sum_x |\text{Stab}(x)|$. If transitive, all stabilizers have same size $|G|/n$ by Orbit-Stabilizer, giving total $n \cdot |G|/n = |G|$. But counting by group elements: $|G|$ from identity, plus contributions from non-identity elements. If each non-identity fixes at least one point, we'd have too many fixed pairs—contradiction. (More rigorous proof requires careful counting.) ∎

5. Compute the conjugacy classes of S_3.

> ***Solution.*** $\text{Cl}(e) = \{e\}$. For (12): conjugating by any permutation gives another transposition, so $\text{Cl}((12)) = \{(12), (13), (23)\}$. For (123): conjugating gives 3-cycles, so $\text{Cl}((123)) = \{(123), (132)\}$. Classes: $\{e\}$, $\{(12), (13), (23)\}$, $\{(123), (132)\}$. ∎

6. Find $Z(S_3)$ and verify the class equation.

> ***Solution.*** An element is in $Z(S_3)$ if it commutes with all elements. Only e commutes with everything, so $Z(S_3) = \{e\}$. Class equation: $|S_3| = |Z(S_3)| + \sum_i [S_3 : C_{S_3}(x_i)]$. From problem 5: $6 = 1 + 3 + 2$. ✓ ∎

7. Let G be a group of order p^2 where p is prime. Prove G is abelian.

> ***Solution.*** By theorem, $|Z(G)| > 1$. So $|Z(G)| \in \{p, p^2\}$. If $|Z(G)| = p^2$, then $Z(G) = G$, so G is abelian. If $|Z(G)| = p$, then $|G/Z(G)| = p$, which is cyclic. But if $G/Z(G)$ is cyclic, then G is abelian (standard result). Either way, G is abelian. ∎

8. Prove that $C_G(x)$ is a subgroup of G.

> ***Solution.*** $C_G(x) = \{g \in G : gx = xg\}$. Identity: $ex = xe$, so $e \in C_G(x)$. Closure: if $g, h \in C_G(x)$, then $(gh)x = g(hx) = g(xh) = (gx)h = (xg)h = x(gh)$, so $gh \in C_G(x)$. Inverses: if $gx = xg$, then $g^{-1}x = g^{-1}xgg^{-1} = g^{-1}gxg^{-1} = xg^{-1}$, so $g^{-1} \in C_G(x)$. ∎

9. Show that if $H \trianglelefteq G$, then H is a union of conjugacy classes.

> ***Solution.*** If $h \in H$ and $H \trianglelefteq G$, then for any $g \in G$, we have $ghg^{-1} \in H$. Thus the entire conjugacy class $\mathrm{Cl}(h) = \{ghg^{-1} : g \in G\}$ is contained in H. Hence $H = \bigcup_{h \in H} \mathrm{Cl}(h)$ is a union of conjugacy classes. ∎

10. Prove that $N_G(H) = \{g \in G : gHg^{-1} = H\}$ is a subgroup of G.

> ***Solution.*** Identity: $eHe^{-1} = H$, so $e \in N_G(H)$. Closure: if $g, h \in N_G(H)$, then $(gh)H(gh)^{-1} = g(hHh^{-1})g^{-1} = gHg^{-1} = H$, so $gh \in N_G(H)$. Inverses: if $gHg^{-1} = H$, then $H = g^{-1}Hg = g^{-1}H(g^{-1})^{-1}$, so $g^{-1} \in N_G(H)$. ∎

11. Use Burnside's Lemma to count the number of distinct ways to color the edges of an equilateral triangle with 3 colors.

> ***Solution.*** D_3 acts on $3^3 = 27$ colorings. Count fixed points for each $g \in D_3$:
>
> - Identity: 27 colorings
> - Rotations by $120^\circ, 240^\circ$: edges must be monochromatic, so 3 colorings each
> - 3 reflections: two edges same color (opposite the axis), so $3^2 = 9$ colorings each
>
> Burnside: $\frac{1}{6}(27 + 3 + 3 + 9 + 9 + 9) = \frac{60}{6} = 10$ distinct colorings. ∎

12. Show that the regular representation of G is faithful.

> ***Solution.*** The regular representation $\varphi : G \to \mathrm{Sym}(G)$ is given by $\varphi(g)(x) = gx$. If $\varphi(g) = \mathrm{id}$, then $gx = x$ for all $x \in G$. In particular, $ge = e$, so $g = e$. Thus $\ker(\varphi) = \{e\}$, and the representation is faithful. ∎

13. Prove that if $|G| = pq$ where $p < q$ are primes and $q \not\equiv 1 \pmod{p}$, then G is cyclic.

> ***Solution.*** By Sylow theory (next chapter), the number n_q of Sylow q-subgroups satisfies $n_q | p$ and $n_q \equiv 1 \pmod{q}$. Since $q > p$, we have $n_q = 1$, so there is a unique (hence normal) Sylow q-subgroup Q of order q. Similarly, $n_p | q$ and $n_p \equiv 1 \pmod{p}$. If $q \not\equiv 1 \pmod{p}$, then $n_p \neq q$, so $n_p = 1$, giving a unique Sylow p-subgroup P. Since P and Q are both normal with $P \cap Q = \{e\}$ and $|PQ| = pq$, we have $G = P \times Q \cong \mathbb{Z}_p \times \mathbb{Z}_q \cong \mathbb{Z}_{pq}$ by Chinese Remainder Theorem. ∎

14. Find $|\mathrm{Orb}(x)|$ and $|\mathrm{Stab}(x)|$ for S_4 acting on unordered pairs from $\{1, 2, 3, 4\}$ by $\sigma \cdot \{i, j\} = \{\sigma(i), \sigma(j)\}$.

> ***Solution.*** The set X of unordered pairs has $\binom{4}{2} = 6$ elements. The action is transitive (any pair can be mapped to any other by a permutation). For $x = \{1, 2\}$: $\mathrm{Stab}(x) = \{\sigma \in S_4 : \{\sigma(1), \sigma(2)\} = \{1, 2\}\}$. This includes permutations that either fix $\{1, 2\}$ setwise: swap $1 \leftrightarrow 2$ or not, and permute $\{3, 4\}$ freely. So $|\mathrm{Stab}(x)| = 2 \cdot 2! = 4$. By Orbit-Stabilizer: $|\mathrm{Orb}(x)| = 24/4 = 6$. ∎

15. Prove that a group of order p^n has a subgroup of order p^k for each $0 \leq k \leq n$.

> ***Solution.*** Proof by induction on n. Base case $n = 0$: only subgroup is $\{e\}$ of order $p^0 = 1$. Inductive step: Assume true for groups of order p^{n-1}. Let $|G| = p^n$. By theorem in this chapter, $|Z(G)| \geq p$. By Cauchy's theorem, $Z(G)$ has an element z of order p. Let $H = \langle z \rangle \trianglelefteq G$ (normal since in center). Then $|G/H| = p^{n-1}$. By induction, G/H has a subgroup K/H of order p^{k-1} for each $1 \leq k \leq n$. Then $K \leq G$ with $|K| = |K/H| \cdot |H| = p^{k-1} \cdot p = p^k$. Together with H (for $k = 1$) and $\{e\}$ (for $k = 0$), we have subgroups of all required orders. ∎

16. Show that every group of order 15 is cyclic.

> ***Solution.*** $15 = 3 \times 5$. Since $5 \not\equiv 1 \pmod{3}$ and $3 < 5$, by problem 13 (or direct argument with Sylow theory), G is cyclic. Alternatively: $n_5 | 3$

> and $n_5 \equiv 1 \pmod 5$ implies $n_5 = 1$. $n_3 | 5$ and $n_3 \equiv 1 \pmod 3$ implies $n_3 \in \{1, 5\}$, but $5 \not\equiv 1 \pmod 3$, so $n_3 = 1$. Both Sylow subgroups normal, so $G \cong \mathbb{Z}_3 \times \mathbb{Z}_5 \cong \mathbb{Z}_{15}$. ∎

17. Compute $|Z(A_4)|$.

> ***Solution.*** A_4 has order 12. Conjugacy classes in A_4 (within A_4, not S_4): $\{e\}$, $\{(12)(34), (13)(24), (14)(23)\}$ (three disjoint 2-cycles), $\{(123), (134), (142), (243)\}$ (one set of 3-cycles), $\{(132), (143), (124), (234)\}$ (another set of 3-cycles). So classes have sizes $1, 3, 4, 4$. An element is in the center iff its conjugacy class is $\{x\}$. Only e has singleton class, so $Z(A_4) = \{e\}$. Thus $|Z(A_4)| = 1$. ∎

18. Let G act on X, and let $x, y \in X$. Prove that if $y = g \cdot x$, then $\text{Stab}(y) = g \cdot \text{Stab}(x) \cdot g^{-1}$.

> ***Solution.*** $h \in \text{Stab}(y)$ iff $h \cdot y = y$ iff $h \cdot (g \cdot x) = g \cdot x$ iff $(g^{-1}hg) \cdot x = x$ iff $g^{-1}hg \in \text{Stab}(x)$ iff $h \in g \cdot \text{Stab}(x) \cdot g^{-1}$. ∎

19. Prove that $N_G(H)$ is the largest subgroup of G in which H is normal.

> ***Solution.*** By definition, $H \trianglelefteq N_G(H)$: for $g \in N_G(H)$ and $h \in H$, we have $ghg^{-1} \in gHg^{-1} = H$. Now suppose $H \trianglelefteq K$ for some $K \leq G$. Then for all $k \in K$, we have $kHk^{-1} = H$, so $k \in N_G(H)$. Thus $K \subseteq N_G(H)$. Therefore $N_G(H)$ is the largest subgroup containing H as normal subgroup. ∎

20. Use Burnside's Lemma to count the number of distinct necklaces with 6 beads, each bead either black or white, where rotations are considered the same.

> ***Solution.*** C_6 acts on $2^6 = 64$ colorings by rotation. Count fixed points for each rotation:
>
> - r^0 (identity): all 64 colorings
> - r^1 (rotate by 60°): pattern repeats every bead, so all same color: 2 colorings

- r^2 (rotate by 120°): pattern repeats every 3 beads: $2^{\gcd(6,2)} = 2^2 = 4$ colorings
- r^3 (rotate by 180°): pattern repeats every 2 beads: $2^{\gcd(6,3)} = 2^3 = 8$ colorings
- r^4 (rotate by 240°): same as r^2: 4 colorings
- r^5 (rotate by 300°): same as r^1: 2 colorings

Burnside: $\frac{1}{6}(64 + 2 + 4 + 8 + 4 + 2) = \frac{84}{6} = 14$ distinct necklaces. ∎

❦

End of Chapter 6

Chapter 7: Sylow Theorems

The Sylow theorems, discovered by Norwegian mathematician Peter Ludwig Mejdell Sylow in 1872, stand among the most elegant and powerful results in finite group theory. They provide a complete analysis of p-subgroups—subgroups whose order is a power of a prime—enabling us to determine the structure of many finite groups.

7.1 p-Groups

> **Definition**
>
> Let p be a prime. A group G is a p-**group** if $|G| = p^n$ for some $n \geq 0$. A subgroup H of G is a p-**subgroup** if H is a p-group.

> **Theorem**
>
> **(Cauchy's Theorem)** If G is a finite group and prime p divides $|G|$, then G has an element of order p.

> ***Proof.*** (Proof by induction on $|G|$.) If $|G| = p$, then G is cyclic and any non-identity element has order p. Assume true for groups of order less than $|G|$, and suppose $p||G|$. Consider two cases:
>
> **Case 1:** There exists a proper subgroup $H < G$ with $p \nmid [G : H]$. Then $|H| = |G|/[G : H]$ is divisible by p (since $p||G|$ but $p \nmid [G : H]$). By induction, H contains an element of order p, hence so does G.
>
> **Case 2:** For every proper subgroup $H < G$, we have $p|[G : H]$. By the class equation, $|G| = |Z(G)| + \sum_i [G : C_G(x_i)]$. Since $C_G(x_i) < G$ for $x_i \notin Z(G)$, we have $p|[G : C_G(x_i)]$. Since $p||G|$ and $p|[G : C_G(x_i)]$ for all terms in the sum, we must have $p||Z(G)|$. Since $Z(G)$ is abelian, it has a subgroup of order p (by induction or by structure theorem for abelian groups), which has an element of order p. ∎

> **Theorem**
>
> If $|G| = p^n$, then $|Z(G)| > 1$ (in fact, $p||Z(G)|$).

> ***Proof.*** Proved in Chapter 6 using the class equation. ∎

If $|G| = p^2$, then G is abelian (hence $G \cong \mathbb{Z}_{p^2}$ or $G \cong \mathbb{Z}_p \times \mathbb{Z}_p$).

> **Example**
>
> Groups of order 4: $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \cong V_4$ (Klein four-group).

Example

Groups of order 9: $\mathbb{Z}_9$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$.

Proposition

Every group of order p^n has a subgroup of order p^k for each $0 \leq k \leq n$.

7.2 Sylow's First Theorem

Definition

Let $|G| = p^n m$ where $p \nmid m$. A **Sylow p-subgroup** of G is a subgroup of order p^n (the maximal power of p dividing $|G|$).

Theorem

(Sylow's First Theorem) Let $|G| = p^n m$ where prime p does not divide m. Then G has a subgroup of order p^k for each $1 \leq k \leq n$. In particular, G has a Sylow p-subgroup.

Proof. We prove existence of a subgroup of order p^n (Sylow p-subgroup). Let G act on $X = \{S \subseteq G : |S| = p^n\}$ by left multiplication: $g \cdot S = \{gs : s \in S\}$. The size of X is $\binom{p^n m}{p^n}$.

Claim: $p \nmid |X|$. This follows from the formula

$$\binom{p^n m}{p^n} = \frac{(p^n m)!}{p^n! \cdot (p^n(m-1))!}.$$

The highest power of p dividing the numerator is higher than that dividing the denominator by exactly the highest power of p dividing m, which is 0 since $p \nmid m$. (Detailed analysis using Legendre's formula confirms this.)

Since $p \nmid |X|$ and $X = \bigcup \mathrm{Orb}(S_i)$, there exists some orbit with $p \nmid |\mathrm{Orb}(S_0)|$. By Orbit-Stabilizer, $|\mathrm{Orb}(S_0)| = |G|/|\mathrm{Stab}(S_0)| = p^n m/|\mathrm{Stab}(S_0)|$. Since $p \nmid |\mathrm{Orb}(S_0)|$, we must have $p^n | |\mathrm{Stab}(S_0)|$. But $|\mathrm{Stab}(S_0)| \leq |S_0| = p^n$, so $|\mathrm{Stab}(S_0)| = p^n$. Thus $\mathrm{Stab}(S_0)$ is a Sylow p-subgroup. ∎

Example

S_3 has order $6 = 2 \cdot 3$. Sylow 2-subgroups: $\langle(12)\rangle$, $\langle(13)\rangle$, $\langle(23)\rangle$ (all order 2). Sylow 3-subgroup: $\langle(123)\rangle = A_3$ (order 3). Note there are three Sylow 2-subgroups but only one Sylow 3-subgroup.

Example

$|A_4| = 12 = 2^2 \cdot 3$. Sylow 2-subgroups have order 4. One is $V_4 = \{e, (12)(34), (13)(24), (14)(23)\}$. This is the unique Sylow 2-subgroup. Sylow 3-subgroups have order 3: $\langle(123)\rangle$, $\langle(124)\rangle$, $\langle(134)\rangle$, $\langle(234)\rangle$ (four of them).

7.3 Sylow's Second Theorem

Theorem

(Sylow's Second Theorem) All Sylow p-subgroups of G are conjugate to each other.

Proof. Let P be a Sylow p-subgroup and Q be any Sylow p-subgroup. Let $X = G/P = \{gP : g \in G\}$ be the set of left cosets. Q acts on X by left multiplication: $q \cdot (gP) = (qg)P$. Since $|X| = [G : P] = m$ where $p \nmid m$, and the orbits partition X, there exists an orbit $\mathrm{Orb}(gP)$ with $p \nmid |\mathrm{Orb}(gP)|$. By Orbit-Stabilizer,

$$|\mathrm{Orb}(gP)| = |Q|/|\mathrm{Stab}_Q(gP)| = p^n/|\mathrm{Stab}_Q(gP)|.$$

Since $p \nmid |\mathrm{Orb}(gP)|$, we have $|\mathrm{Stab}_Q(gP)| = p^n = |Q|$, so $\mathrm{Stab}_Q(gP) = Q$. This means $qgP = gP$ for all $q \in Q$, i.e., $g^{-1}qg \in P$ for all $q \in Q$. Thus $g^{-1}Qg \subseteq P$. Since both have order p^n, we have $g^{-1}Qg = P$, i.e., $Q = gPg^{-1}$. Therefore Q is conjugate to P. ∎

The number of Sylow p-subgroups is $[G : N_G(P)]$ where P is any Sylow p-subgroup and $N_G(P)$ is its normalizer.

> ***Proof.*** The set of Sylow p-subgroups is in bijection with $G/N_G(P)$ via $gN_G(P) \mapsto gPg^{-1}$, since two conjugates gPg^{-1} and hPh^{-1} are equal iff $h^{-1}g \in N_G(P)$ iff $gN_G(P) = hN_G(P)$.

∎

7.4 Sylow's Third Theorem

> **Theorem**
>
> **(Sylow's Third Theorem)** Let n_p denote the number of Sylow p-subgroups. Then:
>
> 1. $n_p \equiv 1 \pmod{p}$
> 2. n_p divides m (where $|G| = p^n m$ with $p \nmid m$)

> ***Proof.*** Let P be a Sylow p-subgroup. Let $\mathcal{S}$ be the set of all Sylow p-subgroups. P acts on $\mathcal{S}$ by conjugation: $p \cdot Q = pQp^{-1}$.
>
> **Fixed points:** $Q \in \mathcal{S}$ is fixed by P iff $pQp^{-1} = Q$ for all $p \in P$, i.e., $P \subseteq N_G(Q)$. Since $Q \trianglelefteq N_G(Q)$ and $P \leq N_G(Q)$, the set $PQ = \{pq : p \in P, q \in Q\}$ is a subgroup with $|PQ| = |P| \cdot |Q|/|P \cap Q| = p^n \cdot p^n/|P \cap Q|$. Since $|PQ|$ divides $|G| = p^n m$, we have $|PQ| \leq p^n m$. But $|PQ| = p^{2n}/|P \cap Q|$ and $p \nmid m$, so $|PQ| \leq p^n$. Combined with $|P|, |Q| \leq |PQ|$ and $|P| = |Q| = p^n$, we get $|PQ| = p^n$, thus $P = Q$. Therefore the only fixed point is P itself.
>
> Since orbits partition $\mathcal{S}$ and orbit sizes divide $|P| = p^n$, non-fixed orbits have size divisible by p. Thus $n_p = |\mathcal{S}| \equiv 1 \pmod{p}$.
>
> For (2): From the corollary to Sylow II, $n_p = [G : N_G(P)]$, which divides $|G| = p^n m$. Since $P \leq N_G(P)$, we have $p^n || N_G(P)|$, so $[G : N_G(P)]$ divides m. ∎

A Sylow p-subgroup P is normal in G if and only if $n_p = 1$.

> ***Proof.*** $P \trianglelefteq G$ iff $N_G(P) = G$ iff $[G : N_G(P)] = 1$ iff $n_p = 1$. ∎

7.5 Applications

Application 1: Groups of Order 15

> **Theorem**
>
> Every group of order 15 is cyclic.

Proof. $15 = 3 \cdot 5$. For Sylow 3-subgroups: $n_3 | 5$ and $n_3 \equiv 1 \pmod{3}$, so $n_3 \in \{1, 5\}$. But $5 \not\equiv 1 \pmod{3}$, so $n_3 = 1$. Similarly, $n_5 | 3$ and $n_5 \equiv 1 \pmod{5}$, so $n_5 = 1$. Let P be the unique Sylow 3-subgroup and Q the unique Sylow 5-subgroup. Both are normal. $P \cap Q = \{e\}$ since $\gcd(3, 5) = 1$. Thus $G = P \times Q \cong \mathbb{Z}_3 \times \mathbb{Z}_5 \cong \mathbb{Z}_{15}$ by Chinese Remainder Theorem. ∎

Application 2: Non-Simplicity of Order 30

Theorem

No group of order 30 is simple.

Proof. $30 = 2 \cdot 3 \cdot 5$. For Sylow 5-subgroups: $n_5 | 6$ and $n_5 \equiv 1 \pmod{5}$. So $n_5 \in \{1, 6\}$. For Sylow 3-subgroups: $n_3 | 10$ and $n_3 \equiv 1 \pmod{3}$. So $n_3 \in \{1, 4, 10\}$. For Sylow 2-subgroups: $n_2 | 15$ and $n_2 \equiv 1 \pmod{2}$ (odd). So $n_2 \in \{1, 3, 5, 15\}$.

If $n_5 = 1$, then the unique Sylow 5-subgroup is normal, so G is not simple. Suppose $n_5 = 6$. Each Sylow 5-subgroup has 4 non-identity elements, and distinct Sylow 5-subgroups intersect trivially. So there are $6 \cdot 4 = 24$ elements of order 5. This leaves $30 - 24 = 6$ elements. If $n_3 = 10$, we'd have $10 \cdot 2 = 20$ elements of order 3, but we only have 6 elements left—contradiction. So $n_3 \in \{1, 4\}$. If $n_3 = 1$, done. If $n_3 = 4$, we have $4 \cdot 2 = 8$ elements of order 3, but we only have 6 elements available—contradiction. Therefore $n_5 = 1$ or $n_3 = 1$, so G has a nontrivial normal subgroup. ∎

Application 3: Classification of Groups of Order pq

Theorem

Let $|G| = pq$ where $p < q$ are primes. If $q \not\equiv 1 \pmod{p}$, then $G \cong \mathbb{Z}_{pq}$.

Proof. $n_q | p$ and $n_q \equiv 1 \pmod{q}$. Since $q > p$, the only possibility is $n_q = 1$. Thus the Sylow q-subgroup Q is normal. $n_p | q$ and $n_p \equiv 1 \pmod{p}$. If $q \not\equiv 1 \pmod{p}$, then $n_p \neq q$, so $n_p = 1$. Thus the Sylow p-subgroup P is also normal. Since $|P| = p$ and $|Q| = q$ with $\gcd(p, q) = 1$, we have $P \cap Q = \{e\}$ and $G = P \times Q \cong \mathbb{Z}_p \times \mathbb{Z}_q \cong \mathbb{Z}_{pq}$. ∎

Example

Groups of order $6 = 2 \cdot 3$: Since $3 \equiv 1 \pmod{2}$, the theorem doesn't apply. Indeed, there are two groups: $\mathbb{Z}_6$ and S_3.

Example

Groups of order $35 = 5 \cdot 7$: Since $7 \not\equiv 1 \pmod{5}$, every group of order 35 is cyclic.

Greek Meditation: Prime Powers and the Atomistic Structure of Groups

The pre-Socratic atomists—Leucippus and Democritus—posited that all matter consists of indivisible atoms moving in void.

Sylow's theorems reveal an analogous atomistic structure in finite groups: every group decomposes into p-primary components, with p-subgroups as the fundamental building blocks. Just as Democritus claimed "By convention sweet, by convention bitter, but in reality atoms and void," we might say: By convention groups have complex structure, but in reality p-subgroups and their interactions. The prime p plays the role of the elementary particle; powers p^n give the atomic weights. The conjugacy of Sylow p-subgroups echoes the interchangeability of atoms of the same type—indistinguishable in their essential nature, differing only in position (conjugacy class). Sylow's theorems show that group structure, though manifold in appearance, reduces to prime-powered essence.

Zen Meditation: The Many Faces of One—Conjugacy as Illusion

Sylow's Second Theorem teaches that all Sylow p-subgroups are conjugate—different in appearance, identical in essence. This is the Zen teaching of *maya* (illusion): the multiplicity of phenomena masks underlying unity. You see many Sylow p-subgroups, but there is only one subgroup—viewed from different perspectives (conjugations). Like the moon reflected in ten thousand pools, or Indra's net where each jewel reflects all others, conjugate subgroups are one and many simultaneously. The koan asks: *If all Sylow p-subgroups are the same, why do we see many?* The answer lies beyond duality. When you realize their conjugacy, distinction collapses—yet the many remain. This is *form is emptiness, emptiness is form*. The count $n_p \equiv 1 \pmod{p}$ hints at primordial unity (the 1) expressing through multiplicity (the p-

> fold manifestation). *Before conjugation, what is the original Sylow subgroup?*

Practice Problems

1. Find all Sylow 2-subgroups and Sylow 3-subgroups of S_3.

> ***Solution.*** $|S_3| = 6 = 2 \cdot 3$. Sylow 2-subgroups (order 2): $\langle(12)\rangle = \{e, (12)\}$, $\langle(13)\rangle = \{e, (13)\}$, $\langle(23)\rangle = \{e, (23)\}$. So $n_2 = 3$. Sylow 3-subgroup (order 3): $A_3 = \langle(123)\rangle = \{e, (123), (132)\}$. So $n_3 = 1$. ∎

2. Verify that $n_3 \equiv 1 \pmod 3$ and $n_3 | 2$ for S_3.

> ***Solution.*** From problem 1, $n_3 = 1$. Check: $1 \equiv 1 \pmod 3$ ✓ and $1 | 2$ ✓. ∎

3. Prove that every group of order 35 is cyclic.

> ***Solution.*** $35 = 5 \cdot 7$. Since $7 > 5$ and $7 = 1 + 6 \not\equiv 1 \pmod 5$, by the theorem in section 7.5, every group of order 35 is isomorphic to $\mathbb{Z}_{35}$, hence cyclic. ∎

4. Show that a group of order 12 has a normal Sylow subgroup.

> ***Solution.*** $12 = 2^2 \cdot 3$. For Sylow 3-subgroups: $n_3 | 4$ and $n_3 \equiv 1 \pmod 3$. So $n_3 \in \{1, 4\}$. For Sylow 2-subgroups: $n_2 | 3$ and $n_2 \equiv 1 \pmod 2$. So $n_2 \in \{1, 3\}$. If $n_3 = 1$, the unique Sylow 3-subgroup is normal, done. If $n_3 = 4$, there are $4 \cdot 2 = 8$ elements of order 3. This leaves $12 - 8 - 1 = 3$

> elements (excluding identity). These must form a Sylow 2-subgroup of order 4. But we can't fit multiple distinct Sylow 2-subgroups of order 4 in just 3 remaining elements (we'd need at least $3 \cdot 3 = 9$ non-identity elements for three such subgroups). So $n_2 = 1$. Either way, some Sylow subgroup is unique hence normal. ∎

5. Prove that no group of order 56 is simple.

> ***Solution.*** $56 = 2^3 \cdot 7$. For Sylow 7-subgroups: $n_7 | 8$ and $n_7 \equiv 1 \pmod 7$. So $n_7 \in \{1, 8\}$. If $n_7 = 1$, done. If $n_7 = 8$, there are $8 \cdot 6 = 48$ elements of order 7. For Sylow 2-subgroups (order 8): $n_2 | 7$ and n_2 odd. So $n_2 \in \{1, 7\}$. If $n_2 = 1$, done. If $n_2 = 7$, each Sylow 2-subgroup contributes at least 4 new elements (rough count), giving at least $7 \cdot 4 = 28$ elements of 2-power order. But $48 + 28 > 56$—contradiction. So $n_2 = 1$ or $n_7 = 1$. ∎

6. Find all possible values of n_5 for a group of order 20.

> ***Solution.*** $20 = 2^2 \cdot 5$. $n_5 | 4$ and $n_5 \equiv 1 \pmod 5$. So $n_5 \in \{1, 2, 4\}$ with $n_5 \equiv 1 \pmod 5$. Check: $1 \equiv 1 \pmod 5$ ✓, $2 \equiv 2 \pmod 5$ ✗, $4 \equiv 4 \pmod 5$ ✗. Thus $n_5 = 1$. ∎

7. Show that if $|G| = p^2 q$ where p, q are distinct primes, then G has a normal Sylow subgroup.

> ***Solution.*** Assume $p < q$ (other case similar). $n_q | p^2$ and $n_q \equiv 1 \pmod q$. So $n_q \in \{1, p, p^2\}$. Since $q > p$, we have $p, p^2 < q$, so $p, p^2 \not\equiv 1 \pmod q$ (they're too small). Thus $n_q = 1$. ∎

8. Classify all groups of order 10.

> ***Solution.*** $10 = 2 \cdot 5$. $n_5 | 2$ and $n_5 \equiv 1 \pmod 5$, so $n_5 = 1$. Thus the Sylow 5-subgroup Q is normal. $n_2 | 5$ and n_2 odd, so $n_2 \in \{1, 5\}$. If $n_2 = 1$, then $G = P \times Q \cong \mathbb{Z}_2 \times \mathbb{Z}_5 \cong \mathbb{Z}_{10}$. If $n_2 = 5$, we have a non-abelian group. Let $P = \langle a \rangle$ (order 2) and $Q = \langle b \rangle$ (order 5). Since $Q \trianglelefteq G$, conjugation by a gives an automorphism of Q. So $aba^{-1} = b^k$ for some k with $\gcd(k, 5) = 1$. Since $a^2 = e$, we have $b = a^2ba^{-2} = a(aba^{-1})a^{-1} = ab^ka^{-1} = b^{k^2}$, so $k^2 \equiv 1 \pmod 5$. Thus $k \in \{1, 4\}$. If $k = 1$, the group is abelian (already covered). If $k = 4 = -1 \pmod 5$, we get $aba^{-1} = b^{-1}$, which gives the dihedral group D_5. Thus: $\mathbb{Z}_{10}$ and D_5. ∎

9. Prove that every group of order 99 is abelian.

> ***Solution.*** $99 = 3^2 \cdot 11$. $n_{11} | 9$ and $n_{11} \equiv 1 \pmod{11}$, so $n_{11} = 1$ (since $9 < 11$). $n_3 | 11$ and $n_3 \equiv 1 \pmod 3$, so $n_3 \in \{1, 11\}$. Check: $11 = 9 + 2 \equiv 2 \pmod 3$ ✗. So $n_3 = 1$. Both Sylow subgroups are normal. Let P (order 9) and Q (order 11). Both are abelian (since $|P| = p^2$ and $|Q|$ is prime). Since they're normal with $P \cap Q = \{e\}$, we have $G = P \times Q$, which is abelian. ∎

10. Show that A_5 is simple (outline using Sylow theorems).

> ***Solution.*** $|A_5| = 60 = 2^2 \cdot 3 \cdot 5$. Check each prime: $n_5 | 12$ and $n_5 \equiv 1 \pmod 5$: $n_5 \in \{1, 6\}$. If $n_5 = 1$, not simple. Compute: $n_5 = 6$ (can verify by counting). $n_3 | 20$ and $n_3 \equiv 1 \pmod 3$: $n_3 \in \{1, 4, 10\}$. Compute: $n_3 = 10$.

$n_2|15$ and n_2 odd: $n_2 \in \{1, 3, 5, 15\}$. Compute: $n_2 = 5$. All $n_p > 1$, so no normal Sylow subgroups. Further analysis of conjugacy classes and normal subgroups shows no nontrivial proper normal subgroups exist. Thus A_5 is simple. ∎

11. Find the number of Sylow 2-subgroups in S_4.

Solution. $|S_4| = 24 = 2^3 \cdot 3$. $n_2|3$ and n_2 odd, so $n_2 \in \{1, 3\}$. Sylow 2-subgroups have order 8. One example: $D_4 \leq S_4$ (symmetries of square). Since S_4 has several such subgroups and they're conjugate, we compute: $n_2 = [S_4 : N_{S_4}(P)]$ where P is a Sylow 2-subgroup. For $P = D_4$, the normalizer has size 8, so $n_2 = 24/8 = 3$. ∎

12. Prove that every group of order pqr (distinct primes $p < q < r$) has a normal Sylow r-subgroup if $r \not\equiv 1 \pmod{q}$ and $r \not\equiv 1 \pmod{p}$.

Solution. $n_r|pq$ and $n_r \equiv 1 \pmod{r}$. So $n_r \in \{1, p, q, pq\}$. Since $r > q > p$, we have $p, q < r$, so $p, q \not\equiv 1 \pmod{r}$. If $pq \equiv 1 \pmod{r}$, then $r|(pq - 1)$. But $pq - 1 < pq < r^2$ (assuming p, q are small enough relative to r). For most cases with $p < q < r$ distinct primes, $pq < r$ or $pq - 1 < r$, so $pq \not\equiv 1 \pmod{r}$. Thus $n_r = 1$ under the given conditions. (Full rigor requires case analysis.) ∎

13. Show that if $|G| = 2p$ where p is an odd prime, then $G \cong \mathbb{Z}_{2p}$ or $G \cong D_p$.

Solution. $n_p|2$ and $n_p \equiv 1 \pmod{p}$, so $n_p = 1$ (since $2 < p$). Thus Sylow p-subgroup Q is normal. $n_2|p$ and n_2 odd. If $n_2 = 1$, then $G = P \times Q \cong \mathbb{Z}_2 \times \mathbb{Z}_p \cong \mathbb{Z}_{2p}$. If $n_2 = p$, let a

> generate P (order 2) and b generate Q (order p). Since $Q \trianglelefteq G$, we have $aba^{-1} = b^k$ for some k. Since $a^2 = e$, we get $k^2 \equiv 1 \pmod{p}$, so $k \in \{1, -1\}$. If $k = 1$, abelian (already covered). If $k = -1$, we have $aba = b^{-1}$, giving relations of D_p. ∎

14. Compute the normalizer $N_{S_3}(P)$ where $P = \langle (12) \rangle$.

> ***Solution.*** $N_{S_3}(P) = \{g \in S_3 : gPg^{-1} = P\}$. Elements: $e, (12) \in N_{S_3}(P)$ trivially. Check others: $(123)(12)(132) = (13) \notin P$, so $(123) \notin N_{S_3}(P)$. Similarly $(132) \notin N_{S_3}(P)$. $(13)(12)(13) = (23) \notin P$, so $(13) \notin N_{S_3}(P)$. Similarly $(23) \notin N_{S_3}(P)$. Thus $N_{S_3}(P) = P = \{e, (12)\}$. ∎

15. Prove that a group of order p^n has a normal subgroup of order p^k for each $0 \leq k \leq n$.

> ***Solution.*** Induction on n. Base: $n = 0$ gives only $\{e\}$. Induction: Assume true for order p^{n-1}. Let $|G| = p^n$. Since $|Z(G)| \geq p$, there exists $z \in Z(G)$ of order p (Cauchy). Let $N = \langle z \rangle \trianglelefteq G$ (normal since in center). Then $|G/N| = p^{n-1}$. By induction, G/N has a normal subgroup K/N of order p^{k-1} for each $1 \leq k \leq n$. Then $K \trianglelefteq G$ with $|K| = |K/N| \cdot |N| = p^{k-1} \cdot p = p^k$. Together with N (for $k = 1$) and $\{e\}$ (for $k = 0$), we have normal subgroups of all orders $p^0, p^1, \ldots, p^n$. ∎

16. Show that if P is a Sylow p-subgroup of G and $P \leq H \leq G$, then P is a Sylow p-subgroup of H.

> ***Solution.*** Let $|G| = p^n m$ with $p \nmid m$, so $|P| = p^n$. Since $H \leq G$, we have $|H| | |G|$. Write $|H| = p^k \cdot m'$ where $p \nmid m'$. Then $k \leq n$. Since $P \leq H$ and $|P| = p^n$, we must have $k \geq n$ (otherwise $|P|$ wouldn't divide $|H|$). Thus $k = n$, so P is a Sylow p-subgroup of H. ∎

17. Find all groups of order 14.

> ***Solution.*** $14 = 2 \cdot 7$. $n_7 | 2$ and $n_7 \equiv 1 \pmod 7$, so $n_7 = 1$. Thus Sylow 7-subgroup Q is normal. $n_2 | 7$ and n_2 odd. If $n_2 = 1$, then $G \cong \mathbb{Z}_2 \times \mathbb{Z}_7 \cong \mathbb{Z}_{14}$. If $n_2 = 7$, we get D_7 (by same analysis as problem 13). Thus: $\mathbb{Z}_{14}$ and D_7. ∎

18. Prove that if $|G| = p^n$ and H is a proper subgroup, then $H \subsetneq N_G(H)$.

> ***Solution.*** G acts on G/H by left multiplication. Since $|G/H| = p^k$ for some $k \geq 1$, and H is the stabilizer of eH, by counting we see that H has at least p cosets. Consider the class equation applied to the action by conjugation on the set of left cosets (or use direct argument): Since $p | |G/H|$, there exists $g \in G$ with $g \notin H$ but $gHg^{-1} = H$ (using the p-group property and induction on structure). Thus $g \in N_G(H) \setminus H$, so $H \subsetneq N_G(H)$. ∎

19. Show that every group of order $255 = 3 \cdot 5 \cdot 17$ is abelian.

> ***Solution.*** Check Sylow subgroups: $n_{17} | 15$ and $n_{17} \equiv 1 \pmod{17}$. Since $15 < 17$, we have $n_{17} = 1$. $n_5 | 51$ and $n_5 \equiv 1 \pmod 5$. Divisors of 51: $1, 3, 17, 51$. Check: $3 \equiv 3, 17 \equiv 2, 51 \equiv 1 \pmod 5$ ✓. Could have $n_5 = 51$. $n_3 | 85$ and $n_3 \equiv 1 \pmod 3$. Could have multiple values.

However, detailed counting of elements (if $n_5 = 51$, then $51 \cdot 4 = 204$ elements of order 5; if $n_3 > 1$, we'd exceed 255) shows both $n_5 = 1$ and $n_3 = 1$. Thus all three Sylow subgroups are normal and cyclic, giving $G \cong \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_{17} \cong \mathbb{Z}_{255}$, which is abelian. ∎

20. Determine all possible orders of Sylow subgroups in a group of order $1176 = 2^3 \cdot 3 \cdot 7^2$.

Solution. Sylow 2-subgroup: order $2^3 = 8$. Sylow 3-subgroup: order 3. Sylow 7-subgroup: order $7^2 = 49$. ∎

❦

End of Chapter 7

Chapter 8: Introduction to Rings

❦

Rings extend groups by introducing a second binary operation—multiplication—that interacts with addition through the distributive law. This additional structure enables us to model arithmetic systems ranging from integers to polynomials to matrices, unifying diverse mathematical objects under a single algebraic framework.

8.1 Ring Axioms and Definition

> **Definition**
>
> A **ring** is a set R together with two binary operations $+$ (addition) and $\cdot$ (multiplication) satisfying:
>
> 1. $(R, +)$ is an abelian group
> 2. Multiplication is associative: $(ab)c = a(bc)$ for all $a, b, c \in R$
> 3. Distributive laws hold:
>   - $a(b + c) = ab + ac$ (left distributivity)
>   - $(a + b)c = ac + bc$ (right distributivity)
>
> If multiplication is commutative, R is a **commutative ring**. If R contains a multiplicative identity 1 (with $1 \cdot a = a \cdot 1 = a$ for all a), we say R is a **ring with unity**.

We denote the additive identity by 0. Unless otherwise stated, we assume rings have unity and $1 \neq 0$.

> **Example**
>
> $(\mathbb{Z}, +, \cdot)$ is a commutative ring with unity. More generally, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are commutative rings with unity.

> **Example**
>
> $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ with addition and multiplication modulo n is a commutative ring with unity.

Example

$M_n(\mathbb{R})$, the set of $n \times n$ real matrices, is a ring with unity (I_n) but is NOT commutative for $n \geq 2$.

Example

$\mathbb{R}[x]$, the set of polynomials with real coefficients, is a commutative ring with unity.

Example

The set $2\mathbb{Z} = \{2k : k \in \mathbb{Z}\}$ of even integers is a commutative ring WITHOUT unity (under usual addition and multiplication).

Proposition

In any ring R:

1. $0 \cdot a = a \cdot 0 = 0$ for all $a \in R$
2. $(-a) \cdot b = a \cdot (-b) = -(ab)$ for all $a, b \in R$
3. $(-a)(-b) = ab$ for all $a, b \in R$
4. If R has unity, then 1 is unique

Proof. (1) $0 \cdot a = (0 + 0) \cdot a = 0 \cdot a + 0 \cdot a$, so subtracting $0 \cdot a$ from both sides gives $0 = 0 \cdot a$. Similarly $a \cdot 0 = 0$.
(2) $ab + (-a)b = (a + (-a))b = 0 \cdot b = 0$, so $(-a)b = -(ab)$.
(3) $(-a)(-b) = -(a(-b)) = -(-(ab)) = ab$ using (2) twice.
(4) If 1 and $1'$ are both multiplicative identities, then $1 = 1 \cdot 1' = 1'$. ∎

8.2 Basic Properties

Definition

Let R be a ring with unity. An element $a \in R$ is a **unit** if there exists $b \in R$ with $ab = ba = 1$. The set of units is denoted $U(R)$ or R^*.

Proposition

$U(R)$ is a group under multiplication.

Proof. Closure: If $a, b \in U(R)$ with inverses a^{-1}, b^{-1}, then $(ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1} = aa^{-1} = 1$, so $ab \in U(R)$ with inverse $b^{-1}a^{-1}$. Identity: $1 \in U(R)$ with $1^{-1} = 1$. Inverses: If $a \in U(R)$, then by definition a^{-1} exists with $aa^{-1} = 1$, and $(a^{-1})^{-1} = a$. ∎

Example

$U(\mathbb{Z}) = \{1, -1\}$.

Example

$U(\mathbb{Z}_n) = \{[a] : \gcd(a, n) = 1\}$ has order $\varphi(n)$ (Euler's totient function).

Example

$U(M_n(\mathbb{R})) = GL_n(\mathbb{R})$, the general linear group of invertible matrices.

Definition

A nonzero element a in a ring R is a **zero divisor** if there exists nonzero $b \in R$ with $ab = 0$ or $ba = 0$.

Example

In $\mathbb{Z}_6$: $[2] \cdot [3] = [6] = [0]$, so both $[2]$ and $[3]$ are zero divisors.

Example

$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ have no zero divisors.

Proposition

A unit cannot be a zero divisor (in a ring with unity).

Proof. Suppose $a \in U(R)$ and $ab = 0$ for some $b \neq 0$. Multiplying by a^{-1}: $a^{-1}(ab) = a^{-1} \cdot 0 = 0$, so $(a^{-1}a)b = b = 0$, contradicting $b \neq 0$. ∎

Definition

An element a is **nilpotent** if $a^n = 0$ for some positive integer n.

Example

In $\mathbb{Z}_9$: $[3]^2 = [9] = [0]$, so $[3]$ is nilpotent.

8.3 Subrings and Ring Extensions

Definition

A subset S of a ring R is a **subring** if S is a ring under the operations inherited from R.

Theorem

(Subring Test) A nonempty subset $S \subseteq R$ is a subring if and only if:

1. $a - b \in S$ for all $a, b \in S$
2. $ab \in S$ for all $a, b \in S$

Proof. ($\Rightarrow$) If S is a subring, it's closed under subtraction and multiplication. ($\Leftarrow$) If conditions hold: (1) implies $(S, +)$ is a subgroup (subgroup test from group theory), and (2) gives closure under multiplication. Associativity and distributivity inherit from R. ∎

Example

$\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$ is a chain of subrings.

Example

$\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$ (Gaussian integers) is a subring of $\mathbb{C}$.

Example

$\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ is a subring of $\mathbb{R}$.

Definition

The **characteristic** of a ring R with unity, denoted $\text{char}(R)$, is the smallest positive integer n such that $n \cdot 1 = \underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} = 0$. If no such n exists, $\text{char}(R) = 0$.

Example

$\text{char}(\mathbb{Z}) = \text{char}(\mathbb{Q}) = \text{char}(\mathbb{R}) = 0$.

Example

$\text{char}(\mathbb{Z}_n) = n$.

Theorem

If R is an integral domain (defined below), then $\text{char}(R)$ is either 0 or prime.

8.4 Integral Domains

Definition

An **integral domain** is a commutative ring with unity that has no zero divisors.

Example

$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are integral domains.

Example

$\mathbb{Z}_p$ is an integral domain if and only if p is prime.

Example

$\mathbb{Z}[i], \mathbb{Z}[\sqrt{2}]$ are integral domains.

Example

$M_n(\mathbb{R})$ for $n \geq 2$ is NOT an integral domain (not commutative, and has zero divisors).

Theorem

(Cancellation Law) In an integral domain R, if $ab = ac$ and $a \neq 0$, then $b = c$.

Proof. $ab = ac$ implies $ab - ac = 0$, so $a(b - c) = 0$. Since R has no zero divisors and $a \neq 0$, we have $b - c = 0$, thus $b = c$. ∎

Theorem

Every finite integral domain is a field (proven below after defining fields).

8.5 Fields

Definition

A **field** is a commutative ring with unity in which every nonzero element is a unit, i.e., $U(F) = F \setminus \{0\}$.

Example

$\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are fields.

Example

$\mathbb{Z}_p$ is a field if and only if p is prime.

Example

$\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ is a field.

Proposition

Every field is an integral domain.

Proof. Suppose F is a field and $ab = 0$ with $a \neq 0$. Since a is a unit, multiply by a^{-1}: $a^{-1}(ab) = a^{-1} \cdot 0 = 0$, so $b = 0$. Thus F has no zero divisors. ∎

Theorem

Every finite integral domain is a field.

Proof. Let D be a finite integral domain, and let $a \in D$ with $a \neq 0$. Define $\phi : D \to D$ by $\phi(x) = ax$. Since D has no zero divisors, ϕ is injective (if $ax = ay$, then $a(x - y) = 0$, so $x = y$). Since D is finite and ϕ is injective, ϕ is surjective. Thus there exists $b \in D$ with $\phi(b) = ab = 1$. Therefore a is a unit, so D is a field. ∎

Definition

For an integral domain R, the **field of fractions** (or **quotient field**) is a field F containing R such that every element of F can be written as a/b with $a, b \in R$ and $b \neq 0$.

Theorem

Every integral domain R can be embedded in its field of fractions F.

Proof. (Sketch) Define $F = \{(a, b) : a, b \in R, b \neq 0\}/\sim$ where $(a, b) \sim (c, d)$ if $ad = bc$. Define addition and multiplication as for rational numbers. This construction generalizes how $\mathbb{Q}$ is constructed from $\mathbb{Z}$. ∎

Example

The field of fractions of $\mathbb{Z}$ is $\mathbb{Q}$.

Example

The field of fractions of $\mathbb{R}[x]$ is $\mathbb{R}(x)$, the field of rational functions.

Greek Meditation: The Ring of Being—Unity and Multiplication in Harmony

Pythagoras proclaimed "All is number," seeing mathematical harmony underlying reality. In rings, we find a richer conception: not merely addition (the accumulation of monads) but multiplication—the interaction and combination of essences. The distributive law, $a(b + c) = ab + ac$, is no mere formal requirement but expresses a profound truth: the whole determines how parts interact. Aristotle's hylomorphism (matter and form) finds algebraic expression: addition gives us matter (the additive group), multiplication gives us form (the multiplicative structure), and their synthesis

via distributivity creates substance (the ring). Zero, the additive identity, represents non-being—*to me on*, that which is not. Unity, when it exists, represents pure being—*to on*, the One from which all emanates. Fields achieve perfection: every non-zero element participates in unity through invertibility, exemplifying Platonic participation in the Form of the Good.

Zen Meditation: Zero and One—The Void and Identity in Ring Structure

Rings enshrine the duality of *śūnyatā* (emptiness) and *tathātā* (suchness). Zero is emptiness—it absorbs all through multiplication ($0 \cdot a = 0$), like the void that contains and negates. One is suchness—it preserves all through multiplication ($1 \cdot a = a$), like the mind that reflects without distorting. Yet zero and one are not-two: both are necessary, neither sufficient alone. The zero divisor presents a koan: $ab = 0$ yet $a, b \neq 0$. How can somethings produce nothing? This shatters our naïve conception of multiplication as "making bigger." The ring teaches: combined, things may annihilate. The field resolves this: eliminate zero divisors, and every element finds its inverse—perfect interdependence, each supporting all. *In the field of fractions, what fraction am I?*

Practice Problems

1. Verify that $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ is a subring of $\mathbb{R}$.

> ***Solution.*** Nonempty: $0 \in \mathbb{Z}[\sqrt{2}]$. Subtraction:
> $(a + b\sqrt{2}) - (c + d\sqrt{2}) = (a - c) + (b - d)\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$. Multiplication:
> $(a + b\sqrt{2})(c + d\sqrt{2}) = ac + ad\sqrt{2} + bc\sqrt{2} + 2bd = (ac + 2bd) + (ad + bc)\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$.
> ∎

2. Find $U(\mathbb{Z}_{12})$.

> ***Solution.*** $U(\mathbb{Z}_{12}) = \{[a] : \gcd(a, 12) = 1\} = \{[1], [5], [7], [11]\}$. Thus
> $|U(\mathbb{Z}_{12})| = \varphi(12) = 4$. ∎

3. Prove that $U(\mathbb{Z}_n)$ is a group under multiplication.

> ***Solution.*** Closure: If $\gcd(a, n) = \gcd(b, n) = 1$, then $\gcd(ab, n) = 1$. Identity: $[1] \in U(\mathbb{Z}_n)$.
> Inverses: If $\gcd(a, n) = 1$, then by Bézout there exist x, y with $ax + ny = 1$, so
> $ax \equiv 1 \pmod{n}$, giving $[a]^{-1} = [x]$. ∎

4. Find all zero divisors in $\mathbb{Z}_{12}$.

> ***Solution.*** Zero divisors: $[a]$ with $\gcd(a, 12) > 1$ and $a \neq 0$. These are
> $[2], [3], [4], [6], [8], [9], [10]$. For example: $[2] \cdot [6] = [12] = [0]$, $[3] \cdot [4] = [12] = [0]$. ∎

5. Prove that $\mathbb{Z}_p$ is an integral domain if and only if p is prime.

> ***Solution.*** ($\Rightarrow$) Suppose $\mathbb{Z}_p$ is an integral domain. If $p = ab$ with $1 < a, b < p$, then $[a][b] = [p] = [0]$ with $[a], [b] \neq [0]$—contradiction. So p is prime. ($\Leftarrow$) If p is prime and $[a][b] = [0]$, then $p|ab$. Since p is prime, $p|a$ or $p|b$, so $[a] = [0]$ or $[b] = [0]$. ∎

6. Show that $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ is a field.

> ***Solution.*** Verify ring axioms (similar to problem 1). For inverses: let $x = a + b\sqrt{2} \neq 0$. Then $x^{-1} = \frac{1}{a+b\sqrt{2}} = \frac{a-b\sqrt{2}}{(a+b\sqrt{2})(a-b\sqrt{2})} = \frac{a-b\sqrt{2}}{a^2-2b^2}$. Since $x \neq 0$, we have $a^2 - 2b^2 \neq 0$ (else $\sqrt{2} = a/b \in \mathbb{Q}$—contradiction). Thus $x^{-1} = \frac{a}{a^2-2b^2} + \frac{-b}{a^2-2b^2}\sqrt{2} \in \mathbb{Q}(\sqrt{2})$. ∎

7. Find $\text{char}(\mathbb{Z}_{15})$ and $\text{char}(\mathbb{Q})$.

> ***Solution.*** $\text{char}(\mathbb{Z}_{15}) = 15$ since $15 \cdot [1] = [15] = [0]$. $\text{char}(\mathbb{Q}) = 0$ since $n \cdot 1 \neq 0$ for any positive integer n. ∎

8. Prove that in a ring with unity, if $a^2 = a$ for all $a \in R$, then R is commutative.

> ***Solution.*** For any $a, b \in R$: $(a + b)^2 = a + b$, so $a^2 + ab + ba + b^2 = a + b$. Since $a^2 = a$ and $b^2 = b$, we get $a + ab + ba + b = a + b$, thus $ab + ba = 0$, so $ab = -ba$. But also $(ab)^2 = ab$, so $ab = -ba = -ab$, giving $2ab = 0$. Similarly, $a^2 = a$ implies $(2a)^2 = 4a^2 = 4a = 2a$, so $4a = 2a$, thus $2a = 0$ for all a. Therefore $ab = -ba = ba$. ∎

9. Show that $\mathbb{Z} \times \mathbb{Z}$ with componentwise operations is a commutative ring with unity that is not an integral domain.

> ***Solution.*** Ring axioms are straightforward. Unity is $(1, 1)$. Zero divisors: $(1, 0) \cdot (0, 1) = (0, 0)$, so not an integral domain. ∎

10. Prove that a finite integral domain is a field using a direct counting argument.

> ***Solution.*** Let D be finite integral domain, $a \in D$, $a \neq 0$. Consider $\{a, a^2, a^3, \ldots\}$. Since D is finite, $a^i = a^j$ for some $i < j$. Then $a^i = a^j$ gives $a^i(1 - a^{j-i}) = 0$. Since no zero divisors and $a^i \neq 0$ (iteratively: $a \neq 0$ and no zero divisors implies $a^n \neq 0$), we have $a^{j-i} = 1$. Thus $a \cdot a^{j-i-1} = 1$, so a is a unit. ∎

11. Find the field of fractions of $\mathbb{Z}[i]$.

> ***Solution.*** $\mathbb{Q}(i) = \{a + bi : a, b \in \mathbb{Q}\}$, the field of Gaussian rationals. Any element can be written as $\frac{p+qi}{r+si}$ where $p, q, r, s \in \mathbb{Z}$ and $r + si \neq 0$, which simplifies to Gaussian rational form by multiplying by conjugate. ∎

12. Prove that if R is an integral domain and $\text{char}(R) = n > 0$, then n is prime.

> ***Solution.*** Suppose $n = ab$ with $1 < a, b < n$. Then $n \cdot 1 = (ab) \cdot 1 = (a \cdot 1)(b \cdot 1) = 0$. Since R is an integral domain, either $a \cdot 1 = 0$ or $b \cdot 1 = 0$. But this contradicts minimality of n as the characteristic. Thus n is prime. ∎

13. Show that $U(\mathbb{Z} \times \mathbb{Z}) = \{(1,1),(1,-1),(-1,1),(-1,-1)\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

> ***Solution.*** $(a,b) \in U(\mathbb{Z} \times \mathbb{Z})$ iff there exists (c,d) with $(a,b)(c,d) = (1,1)$, i.e., $ac = 1$ and $bd = 1$. Since $U(\mathbb{Z}) = \{\pm 1\}$, we need $a, b \in \{\pm 1\}$. Thus $U(\mathbb{Z} \times \mathbb{Z}) = \{\pm 1\} \times \{\pm 1\}$. This has order 4 with all non-identity elements of order 2, so $\cong \mathbb{Z}_2 \times \mathbb{Z}_2$. ∎

14. Find all nilpotent elements in $\mathbb{Z}_{36}$.

> ***Solution.*** $[a]^n = [0]$ means $36 | a^n$. Since $36 = 2^2 \cdot 3^2$, we need $4|a^n$ and $9|a^n$. For $4|a^n$: need $2|a$. For $9|a^n$: need $3|a$. So $6|a$. Nilpotent elements: $[0], [6], [12], [18], [24], [30]$. Check: $[6]^2 = [36] = [0]$. ∎

15. Prove that if F is a field, then $F[x]$ (polynomials over F) is an integral domain.

> ***Solution.*** $F[x]$ is commutative ring with unity. For zero divisors: suppose $f(x)g(x) = 0$ with $f, g \neq 0$. Let $\deg(f) = n$, $\deg(g) = m$ with leading coefficients $a_n, b_m \neq 0$. Then $\deg(fg) = n + m$ with leading coefficient $a_n b_m$. Since F is a field (hence integral domain), $a_n b_m \neq 0$, so $fg \neq 0$—contradiction. Thus no zero divisors. ∎

16. Show that the intersection of two subrings is a subring.

> ***Solution.*** Let S, T be subrings of R. Then $S \cap T \neq \emptyset$ (contains 0). If $a, b \in S \cap T$: $a - b \in S$ (since S subring) and $a - b \in T$ (since T subring), so $a - b \in S \cap T$. Similarly $ab \in S \cap T$. By subring test, $S \cap T$ is a subring. ∎

17. Find $U(M_2(\mathbb{Z}_2))$.

> ***Solution.*** $U(M_2(\mathbb{Z}_2)) = GL_2(\mathbb{Z}_2)$, matrices with nonzero determinant over $\mathbb{Z}_2$. Matrices in $M_2(\mathbb{Z}_2)$: $2^4 = 16$ total. Det $\neq 0$ means columns are linearly independent: 6 choices for first column (nonzero), 6 for second (not a scalar multiple of first, but over $\mathbb{Z}_2$ this means not equal), so $3 \cdot 2 = 6$ matrices. Actually: $|GL_2(\mathbb{Z}_2)| = (2^2 - 1)(2^2 - 2) = 3 \cdot 2 = 6$. ∎

18. Prove that every subring of a field that contains the unity is an integral domain.

> ***Solution.*** Let F be a field and R a subring containing 1. Suppose $ab = 0$ in R with $a \neq 0$. Then $ab = 0$ in F as well. Since F is a field (hence integral domain), $b = 0$. Thus R has no zero divisors. Since R inherits commutativity and unity from F, R is an integral domain. ∎

19. Show that if R has unity and characteristic p (prime), then $(a+b)^p = a^p + b^p$ for all $a, b \in R$.

> ***Solution.*** By binomial theorem: $(a+b)^p = \sum_{k=0}^{p} \binom{p}{k} a^{p-k} b^k$. For $0 < k < p$: $\binom{p}{k} = \frac{p!}{k!(p-k)!}$ is divisible by p (since p is prime and doesn't divide $k!$ or $(p-k)!$ for $0 < k < p$). In characteristic p, $p \cdot 1 = 0$, so $\binom{p}{k} \cdot 1 = 0$. Thus all middle terms vanish, leaving $(a+b)^p = a^p + b^p$. ∎

20. Construct a field with 4 elements.

Solution. $\mathbb{F}_4 = \mathbb{Z}_2[x]/(x^2+x+1)$. Elements: $\{0, 1, \alpha, \alpha+1\}$ where α is a root of x^2+x+1, so $\alpha^2 = \alpha + 1$. Addition and multiplication tables can be computed. Alternatively: $\{0, 1, \alpha, \beta\}$ with $\alpha^2 = \beta$, $\alpha\beta = 1+\alpha$, $\beta^2 = 1+\beta$, etc. (Requires careful verification of field axioms.) ∎

❦

End of Chapter 8

Chapter 9: Ideals and Quotient Rings

❦

Just as normal subgroups enable the construction of quotient groups, ideals—special subrings that absorb multiplication—enable quotient rings. The parallel between group theory and ring theory deepens with the isomorphism theorems, revealing the fundamental unity of algebraic structure.

9.1 Ideals

> **Definition**
>
> A subset I of a ring R is a **left ideal** if:
>
> 1. $(I, +)$ is a subgroup of $(R, +)$
> 2. $ra \in I$ for all $r \in R$ and $a \in I$ (absorption property)
>
> A **right ideal** satisfies $ar \in I$ instead. A **two-sided ideal** (or simply **ideal**) is both a left and right ideal.

For commutative rings, all three notions coincide. Unless stated otherwise, "ideal" means two-sided ideal.

> **Example**
>
> In any ring R: $\{0\}$ and R are ideals (trivial ideals).

> **Example**
>
> In $\mathbb{Z}$: $n\mathbb{Z} = \{nk : k \in \mathbb{Z}\}$ is an ideal for any $n \in \mathbb{Z}$.

> **Example**
>
> In $F[x]$ (polynomials over field F): the set of polynomials divisible by $x^2 + 1$ is an ideal.

> **Example**
>
> In $M_n(\mathbb{R})$: the set of matrices with first row zero is a left ideal but not a right ideal (for $n \geq 2$).

Proposition

If I is an ideal of R and $1 \in I$, then $I = R$.

Proof. For any $r \in R$: $r = r \cdot 1 \in I$ (by absorption). Thus $I = R$. ∎

Definition

An ideal I is **proper** if $I \neq R$.

Proposition

An ideal I is proper if and only if I contains no units.

Proof. If $u \in I$ is a unit, then $1 = u \cdot u^{-1} \in I$, so $I = R$. Conversely, if $I = R$, then $1 \in I$. ∎

Definition

The ideal **generated** by a set $S \subseteq R$, denoted $\langle S \rangle$ or (S), is the smallest ideal containing S:

$$\langle S \rangle = \left\{ \sum_{i=1}^{n} r_i s_i r_i' : n \in \mathbb{N}, r_i, r_i' \in R, s_i \in S \right\}.$$

For commutative rings: $\langle S \rangle = \{r_1 s_1 + \cdots + r_n s_n : r_i \in R, s_i \in S\}$.

Definition

An ideal I is **principal** if $I = \langle a \rangle$ for some $a \in R$. We write $\langle a \rangle = (a) = Ra = \{ra : r \in R\}$ (for commutative rings).

Example

Every ideal in $\mathbb{Z}$ is principal: $I = n\mathbb{Z} = \langle n \rangle$ for some $n \geq 0$.

Proposition

If I and J are ideals of R, then:

1. $I \cap J$ is an ideal
2. $I + J = \{a + b : a \in I, b \in J\}$ is an ideal
3. $IJ = \{\sum a_i b_i : a_i \in I, b_i \in J\}$ is an ideal

9.2 Quotient Rings

Definition

Let I be an ideal of R. The **quotient ring** R/I is the set of cosets $\{r + I : r \in R\}$ with operations:

- $(r + I) + (s + I) = (r + s) + I$
- $(r + I)(s + I) = rs + I$

Theorem

If I is an ideal of R, then R/I is a ring under the operations defined above.

Proof. First verify operations are well-defined. Suppose $r + I = r' + I$ and $s + I = s' + I$. Then $r' = r + i_1$ and $s' = s + i_2$ for some $i_1, i_2 \in I$.

Addition: $(r' + s') + I = (r + i_1 + s + i_2) + I = (r + s) + I$ since $i_1 + i_2 \in I$.

Multiplication: $r's' + I = (r + i_1)(s + i_2) + I = rs + ri_2 + i_1 s + i_1 i_2 + I$. Since I is an ideal, $ri_2, i_1 s, i_1 i_2 \in I$, so $r's' + I = rs + I$.

Ring axioms: $(R/I, +)$ is an abelian group (quotient group from group theory). Associativity of multiplication and distributivity inherit from R. Additive identity is $0 + I = I$. If R has unity 1, then $1 + I$ is the unity of R/I. ∎

Example

$\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$ (the integers modulo n).

Example

$\mathbb{R}[x]/(x^2 + 1) \cong \mathbb{C}$. The coset $x + (x^2 + 1)$ behaves like i since $x^2 + 1 = 0$ in the quotient, i.e., $x^2 = -1$.

Example

$\mathbb{Q}[x]/(x^2 - 2) \cong \mathbb{Q}(\sqrt{2})$. The coset $x + (x^2 - 2)$ represents $\sqrt{2}$.

9.3 Ring Homomorphisms

Definition

A **ring homomorphism** is a function $\varphi : R \to S$ between rings satisfying:

1. $\varphi(a+b) = \varphi(a) + \varphi(b)$ for all $a, b \in R$
2. $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in R$
3. $\varphi(1_R) = 1_S$ (if rings have unity)

Definition

The **kernel** of $\varphi : R \to S$ is $\ker(\varphi) = \{r \in R : \varphi(r) = 0\}$. The **image** is $\mathrm{Im}(\varphi) = \{\varphi(r) : r \in R\}$.

Proposition

Let $\varphi : R \to S$ be a ring homomorphism. Then:

1. $\ker(\varphi)$ is an ideal of R
2. $\mathrm{Im}(\varphi)$ is a subring of S
3. φ is injective if and only if $\ker(\varphi) = \{0\}$

Proof. (1) $\ker(\varphi)$ is a subgroup of $(R, +)$ (from group theory). For absorption: if $a \in \ker(\varphi)$ and $r \in R$, then $\varphi(ra) = \varphi(r)\varphi(a) = \varphi(r) \cdot 0 = 0$, so $ra \in \ker(\varphi)$. Similarly $ar \in \ker(\varphi)$.
(2) Straightforward verification.
(3) Same as for group homomorphisms. ∎

Example

The evaluation homomorphism $\varphi : \mathbb{R}[x] \to \mathbb{R}$ defined by $\varphi(f(x)) = f(0)$ has kernel $\langle x \rangle = x\mathbb{R}[x]$ (polynomials with constant term 0).

Example

The natural projection $\pi : R \to R/I$ defined by $\pi(r) = r + I$ is a surjective ring homomorphism with $\ker(\pi) = I$.

9.4 Isomorphism Theorems for Rings

Theorem

(First Isomorphism Theorem) Let $\varphi : R \to S$ be a ring homomorphism. Then $R/\ker(\varphi) \cong \mathrm{Im}(\varphi)$.

Proof. Define $\psi : R/\ker(\varphi) \to \mathrm{Im}(\varphi)$ by $\psi(r + \ker(\varphi)) = \varphi(r)$.

Well-defined: If $r + \ker(\varphi) = s + \ker(\varphi)$, then $r - s \in \ker(\varphi)$, so $\varphi(r - s) = 0$, thus $\varphi(r) = \varphi(s)$.

Homomorphism:
$\psi((r + \ker(\varphi)) + (s + \ker(\varphi))) = \psi((r + s) + \ker(\varphi)) = \varphi(r + s) = \varphi(r) + \varphi(s) = \psi(r +$. Similarly for multiplication.

Injective: $\psi(r + \ker(\varphi)) = 0$ iff $\varphi(r) = 0$ iff $r \in \ker(\varphi)$ iff $r + \ker(\varphi) = \ker(\varphi)$ (the zero element).

Surjective: For any $\varphi(r) \in \mathrm{Im}(\varphi)$, we have $\psi(r + \ker(\varphi)) = \varphi(r)$. ∎

Example

Consider $\varphi : \mathbb{Z} \to \mathbb{Z}_n$ with $\varphi(a) = [a]$. Then $\ker(\varphi) = n\mathbb{Z}$, so $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$.

Theorem

(Second Isomorphism Theorem) Let I and J be ideals of R. Then $(I + J)/J \cong I/(I \cap J)$.

Theorem

(Third Isomorphism Theorem) Let $I \subseteq J$ be ideals of R. Then $(R/I)/(J/I) \cong R/J$.

Theorem

(Correspondence Theorem) Let I be an ideal of R. There is a bijection between ideals of R containing I and ideals of R/I, given by $J \mapsto J/I$.

9.5 Prime and Maximal Ideals

Definition

A proper ideal P of a commutative ring R is **prime** if whenever $ab \in P$, either $a \in P$ or $b \in P$.

Definition

A proper ideal M of a commutative ring R is **maximal** if there is no ideal I with $M \subsetneq I \subsetneq R$.

Theorem

Let R be a commutative ring with unity and I a proper ideal. Then:

1. I is prime if and only if R/I is an integral domain
2. I is maximal if and only if R/I is a field

Proof. (1) (⇒) Suppose I is prime. R/I is commutative with unity $1 + I$. If $(a + I)(b + I) = I$ (the zero element), then $ab + I = I$, so $ab \in I$. Since I is prime, $a \in I$ or $b \in I$, i.e., $a + I = I$ or $b + I = I$. Thus R/I has no zero divisors, so it's an integral domain.
(⇐) Suppose R/I is an integral domain. If $ab \in I$, then $(a + I)(b + I) = ab + I = I$. Since R/I has no zero divisors, $a + I = I$ or $b + I = I$, so $a \in I$ or $b \in I$. Thus I is prime.

(2) (⇒) Suppose I is maximal. R/I is a commutative ring with unity. Let $a + I \neq I$ in R/I, so $a \notin I$. Consider $J = I + \langle a \rangle = \{i + ra : i \in I, r \in R\}$. This is an ideal with $I \subsetneq J$ (since $a \in J \setminus I$). By maximality, $J = R$. Thus $1 \in J$, so $1 = i + ra$ for some $i \in I, r \in R$. Then $1 + I = ra + I = (r + I)(a + I)$, so $a + I$ has inverse $r + I$. Thus R/I is a field.
(⇐) Suppose R/I is a field. Let J be an ideal with $I \subseteq J \subseteq R$. If $J \neq I$, pick $a \in J \setminus I$. Then $a + I \neq I$ in R/I, so $a + I$ has an inverse: $(a + I)(b + I) = 1 + I$ for some b. Thus $ab - 1 \in I \subseteq J$. Since $a \in J$ and $ab \in J$ (ideal property), we have $ab - 1 \in J$ and $ab \in J$, so $1 = ab - (ab - 1) \in J$. Thus $J = R$. Therefore I is maximal. ∎

Every maximal ideal is prime (since every field is an integral domain).

Example

In $\mathbb{Z}$: $p\mathbb{Z}$ is maximal (and prime) if and only if p is prime. For $\mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z}_p$ is a field iff p is prime.

Example

In $\mathbb{R}[x]$: $\langle x^2 + 1 \rangle$ is maximal since $\mathbb{R}[x]/(x^2 + 1) \cong \mathbb{C}$ is a field.

Example

In $\mathbb{Z}[x]$: $\langle 2, x \rangle$ is maximal (quotient is $\mathbb{Z}_2$). Also $\langle 2 \rangle$ is prime but not maximal.

Greek Meditation: Ideals as Shadows of the Perfect—Plato's Cave in Ring Theory

In Plato's allegory of the cave, prisoners see only shadows of true Forms. Ideals play a similar role: they are "shadows" or "defects" within the ring—elements that, when combined with the ring via multiplication, remain trapped within the ideal. The quotient R/I is the ascent from cave to sunlight: we collapse the ideal to zero, revealing the true structure freed from shadow. Maximal ideals correspond to the highest Forms—when we quotient by a maximal ideal, we achieve a field, the most perfect ring structure where every nonzero element has an inverse (full participation in Being). Prime ideals, slightly less restrictive, still embody a kind of purity: the quotient is an integral domain (no zero divisors—no fundamental contradictions). The isomorphism theorems are the ladder between cave levels: they show that different perspectives (quotients at different stages) are isomorphic—equal in form though not in matter. As Heraclitus taught that the way up and the way down are one and the same, the correspondence theorem reveals that structure ascends and descends between R and R/I in bijective harmony.

Zen Meditation: The Quotient Mind—Collapsing Distinctions in Rings

To form R/I, we declare: all elements differing by elements of I are the same. This is the Zen practice of non-discrimination. The ideal I represents attachments, concepts, distinctions we release. When we quotient, we see through these illusions to a simpler reality. The kernel of a homomorphism is precisely what must be relinquished (sent to zero) for the map to be well-defined—echoing the Buddhist teaching that attachment is the root of suffering. The First Isomorphism Theorem states: once you let go of the kernel, what remains is isomorphic to the image—form and emptiness are not-two. Maximal ideals point to ultimate simplicity: quotient by a maximal ideal, and you reach a field—a structure of perfect symmetry where every question (division problem) has an answer. This is enlightenment: the quotient by all ignorance (maximal ideal) reveals the Buddha-nature (field structure). *What remains when you quotient your mind by the ideal of self?*

Practice Problems

1. Verify that $2\mathbb{Z}$ is an ideal of $\mathbb{Z}$.

> ***Solution.*** $(2\mathbb{Z}, +)$ is a subgroup of $(\mathbb{Z}, +)$. For absorption: if $a \in 2\mathbb{Z}$ (so $a = 2k$) and $r \in \mathbb{Z}$, then $ra = r(2k) = 2(rk) \in 2\mathbb{Z}$. ∎

2. Show that the set of matrices in $M_2(\mathbb{R})$ with zero first row is a left ideal but not a right ideal.

> ***Solution.*** Let $I = \left\{ \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} : a, b \in \mathbb{R} \right\}$. $(I, +)$ is subgroup. For left absorption: $\begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} = \begin{pmatrix} ya & yb \\ za + wa & zb + wb \end{pmatrix}$, which may not have zero first row. Wait—let me recalculate: $\begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ ax + bz & ay + bw \end{pmatrix} \in I$. So it's a right ideal. For left: $\begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} = \begin{pmatrix} ay & by \\ ay & by \end{pmatrix}$—wait, recalculate: $= \begin{pmatrix} 0 \cdot x + 0 \cdot z & 0 \cdot y + 0 \cdot w \\ a \cdot x + b \cdot z & a \cdot y + b \cdot w \end{pmatrix}$. Actually, first row is zero in the ideal element. So: matrix multiply from left: $\begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} = \begin{pmatrix} ya & yb \\ wa & wb \end{pmatrix} \notin I$ generally. So it's a right ideal but not left ideal. ∎

3. Prove that every ideal in $\mathbb{Z}$ is principal.

> ***Solution.*** Let I be an ideal in $\mathbb{Z}$. If $I = \{0\}$, then $I = \langle 0 \rangle$. Otherwise, let n be the smallest positive integer in I. For any $a \in I$, write $a = qn + r$ with $0 \leq r < n$ (division algorithm). Since $a, qn \in I$, we have $r = a - qn \in I$. By minimality of n and $0 \leq r < n$, we must have $r = 0$. Thus $a = qn \in \langle n \rangle$. Therefore $I = n\mathbb{Z} = \langle n \rangle$. ∎

4. Find $\ker(\varphi)$ for the evaluation homomorphism $\varphi : \mathbb{R}[x] \to \mathbb{R}$ with $\varphi(f(x)) = f(2)$.

> ***Solution.*** $\ker(\varphi) = \{f(x) \in \mathbb{R}[x] : f(2) = 0\} = \langle x - 2 \rangle$, the ideal of polynomials with 2 as a root. ∎

5. Prove that $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}_6$.

> ***Solution.*** Define $\varphi : \mathbb{Z} \to \mathbb{Z}_6$ by $\varphi(n) = [n]$. This is a surjective ring homomorphism with $\ker(\varphi) = 6\mathbb{Z}$. By First Isomorphism Theorem, $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}_6$. ∎

6. Show that $\langle 2 \rangle$ is a prime ideal in $\mathbb{Z}$ but not maximal.

> ***Solution.*** $\mathbb{Z}/\langle 2 \rangle \cong \mathbb{Z}_2$, which is an integral domain (in fact, a field since 2 is prime), so $\langle 2 \rangle$ is prime. But $\mathbb{Z}_2$ is a field, so actually $\langle 2 \rangle$ IS maximal. (Correction: for any prime p, $\langle p \rangle$ is maximal in $\mathbb{Z}$.) ∎

7. Prove that $\mathbb{R}[x]/(x^2 + 1) \cong \mathbb{C}$.

> ***Solution.*** Define $\varphi : \mathbb{R}[x] \to \mathbb{C}$ by $\varphi(f(x)) = f(i)$. This is a surjective ring homomorphism. $\ker(\varphi) = \{f(x) : f(i) = 0\} = \langle x^2 + 1 \rangle$ (since $x^2 + 1$ is irreducible over $\mathbb{R}$ and has i as root). By First Isomorphism Theorem, $\mathbb{R}[x]/(x^2 + 1) \cong \mathbb{C}$. ∎

8. Determine whether $\langle x \rangle$ is a prime ideal in $\mathbb{Z}[x]$.

> ***Solution.*** $\mathbb{Z}[x]/\langle x \rangle \cong \mathbb{Z}$ (evaluation at $x = 0$). Since $\mathbb{Z}$ is an integral domain, $\langle x \rangle$ is prime. ∎

9. Show that $\langle x \rangle$ is not a maximal ideal in $\mathbb{Z}[x]$.

> ***Solution.*** $\mathbb{Z}[x]/\langle x\rangle \cong \mathbb{Z}$, which is not a field (e.g., 2 has no multiplicative inverse). So $\langle x\rangle$ is not maximal. Alternatively: $\langle x\rangle \subsetneq \langle 2, x\rangle \subsetneq \mathbb{Z}[x]$ is a proper containment. ∎

10. Prove that if I and J are ideals with $I + J = R$, then $I \cap J = IJ$.

> ***Solution.*** Always $IJ \subseteq I \cap J$ (since $IJ \subseteq I$ and $IJ \subseteq J$). For reverse: since $I + J = R$, we have $1 = a + b$ for some $a \in I, b \in J$. For any $x \in I \cap J$: $x = x \cdot 1 = x(a + b) = xa + xb$. Since $x, a \in I$, we have $xb \in IJ$. Since $x, b \in J$, we have $xa \in IJ$. Thus $x = xa + xb \in IJ$. ∎

11. Find all ideals of $\mathbb{Z}_6$.

> ***Solution.*** Ideals correspond to subgroups under addition: $\{[0]\}$, $\{[0], [3]\}$, $\{[0], [2], [4]\}$, $\mathbb{Z}_6$. Must verify ideal property (absorption), which holds for all since $\mathbb{Z}_6$ is commutative. ∎

12. Prove that the intersection of all prime ideals containing an ideal I is a well-defined object (called the radical of I).

> ***Solution.*** At minimum, R itself might be considered (though we usually require proper ideals). For any ideal $I \neq R$, Zorn's Lemma guarantees existence of maximal ideals containing I, and every maximal ideal is prime. So the intersection is over a nonempty set of ideals. The intersection of ideals is an ideal (proven earlier). ∎

13. Show that in $\mathbb{Z}[i]$, the ideal $\langle 1 + i\rangle$ is prime.

> ***Solution.*** $\mathbb{Z}[i]/\langle 1 + i\rangle \cong \mathbb{Z}_2$. To see this: define $\varphi : \mathbb{Z}[i] \to \mathbb{Z}_2$ by $\varphi(a + bi) = [a + b] \pmod 2$. Check homomorphism:
> $\varphi((a + bi)(c + di)) = \varphi((ac - bd) + (ad + bc)i) = [ac - bd + ad + bc] = [ac + bd + ad$ - ... Actually easier: $\varphi(1 + i) = [1 + 1] = [0]$, so $\langle 1 + i\rangle \subseteq \ker(\varphi)$. One can verify $\ker(\varphi) = \langle 1 + i\rangle$ and surjectivity, giving $\mathbb{Z}[i]/\langle 1 + i\rangle \cong \mathbb{Z}_2$. Since $\mathbb{Z}_2$ is an integral domain, $\langle 1 + i\rangle$ is prime. ∎

14. Describe the quotient ring $\mathbb{Q}[x]/(x^2 - 5)$.

> ***Solution.*** $\mathbb{Q}[x]/(x^2 - 5) \cong \mathbb{Q}(\sqrt{5})$, the field $\{a + b\sqrt{5} : a, b \in \mathbb{Q}\}$. In the quotient, $x^2 = 5$, so x behaves like $\sqrt{5}$. ∎

15. Prove that if R is a field, then the only ideals are $\{0\}$ and R.

> ***Solution.*** Let I be a nonzero ideal. Pick $0 \neq a \in I$. Since R is a field, a is a unit, so a^{-1} exists. Thus $1 = a \cdot a^{-1} \in I$ (by absorption). Therefore $I = R$. ∎

16. Find a maximal ideal in $\mathbb{Z}[x]$.

> ***Solution.*** $\langle 2, x\rangle$ is maximal. $\mathbb{Z}[x]/\langle 2, x\rangle \cong \mathbb{Z}_2$ (polynomials with coefficients mod 2 and $x = 0$), which is a field. ∎

17. Show that if $\varphi : R \to S$ is a ring homomorphism and J is an ideal of S, then $\varphi^{-1}(J)$ is an ideal of R.

> ***Solution.*** $\varphi^{-1}(J) = \{r \in R : \varphi(r) \in J\}$. Subgroup property: if $a, b \in \varphi^{-1}(J)$, then $\varphi(a-b) = \varphi(a) - \varphi(b) \in J$, so $a - b \in \varphi^{-1}(J)$. Absorption: if $a \in \varphi^{-1}(J)$ and $r \in R$, then $\varphi(ra) = \varphi(r)\varphi(a) \in J$ (since J is ideal and $\varphi(a) \in J$), so $ra \in \varphi^{-1}(J)$. ∎

18. Prove the Third Isomorphism Theorem for rings.

> ***Solution.*** Let $I \subseteq J$ be ideals of R. Define $\varphi : R/I \to R/J$ by $\varphi(r + I) = r + J$. This is well-defined surjective ring homomorphism with $\ker(\varphi) = J/I$. By First Isomorphism Theorem, $(R/I)/\ker(\varphi) \cong R/J$, i.e., $(R/I)/(J/I) \cong R/J$. ∎

19. Show that in a principal ideal domain (PID), every nonzero prime ideal is maximal.

> ***Solution.*** Let P be a nonzero prime ideal in PID R, so $P = \langle p \rangle$ for some p. Suppose M is an ideal with $P \subseteq M \subseteq R$. Then $M = \langle m \rangle$ for some m. Since $p \in P \subseteq M$, we have $p = rm$ for some $r \in R$. Since P is prime and $rm = p \in P$, either $r \in P$ or $m \in P$. If $m \in P = \langle p \rangle$, then $m = sp$, so $p = rm = r(sp) = (rs)p$, giving $1 = rs$ (cancellation), so r is a unit, thus $M = \langle m \rangle = \langle sp \rangle = \langle p \rangle = P$. If $r \in P = \langle p \rangle$, then $r = tp$, so $p = rm = (tp)m$, giving $1 = tm$ (cancellation), so m is a unit, thus $M = \langle m \rangle = R$. Therefore $M = P$ or $M = R$, so P is maximal. ∎

20. Determine all ring homomorphisms from $\mathbb{Z}_{12}$ to $\mathbb{Z}_8$.

> ***Solution.*** A ring homomorphism $\varphi : \mathbb{Z}_{12} \to \mathbb{Z}_8$ is determined by $\varphi([1])$ (since $[1]$ generates $\mathbb{Z}_{12}$ additively). Let $\varphi([1]) = [k]$. Then $[0] = \varphi([0]) = \varphi([12]) = 12\varphi([1]) = [12k]$ in $\mathbb{Z}_8$. So $8 | 12k$, i.e., $2 | 3k$. Since $\gcd(2,3) = 1$, we need $2 | k$. Also $\varphi([1])^2 = \varphi([1]^2) = \varphi([1])$ (since $[1] \cdot [1] = [1]$ and must preserve multiplication: $[k]^2 = [k]$ in $\mathbb{Z}_8$). Wait, that's not quite right. Let me reconsider: φ must preserve unity, so $\varphi([1]) = [1]$ (if considering unital rings). Then $\varphi([n]) = [n]$ for all n, but need well-defined: if $n \equiv m \pmod{12}$, then $\varphi([n]) = \varphi([m])$ automatically. But does $\varphi([n]) = [n]$ make sense in $\mathbb{Z}_8$? We need to check homomorphism. Actually, there's an issue: $\varphi([1])$ must satisfy $12\varphi([1]) = 0$ in $\mathbb{Z}_8$, and if φ is unital, $\varphi([1]) = [1]$, but $12[1] = [12] = [4] \neq [0]$ in $\mathbb{Z}_8$. So there is NO unital ring homomorphism. If we drop unity requirement: $\varphi([1]) = [k]$ with $[12k] = [0]$, so $k \in \{0, 2, 4, 6\}$. Each gives a homomorphism (verify). ∎

End of Chapter 9

Chapter 10: Polynomial Rings

❦

Polynomials form the bridge between abstract ring theory and concrete algebraic equations, encoding the fundamental mysteries of solvability, factorization, and field extensions. The ring of polynomials over a field possesses remarkable structural properties—principal ideal domains where unique factorization reigns—making them foundational to both classical and modern algebra.

10.1 Polynomial Rings

> **Definition**
>
> Let R be a commutative ring with unity. A **polynomial** over R is a formal expression
>
> $$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = \sum_{i=0}^{n} a_i x^i$$
>
> where $a_i \in R$ are called **coefficients**, and x is an **indeterminate**. The set of all polynomials over R, denoted $R[x]$, forms the **polynomial ring** over R.

Two polynomials are equal if and only if all corresponding coefficients are equal. For $f(x) = \sum_{i=0}^{n} a_i x^i$ with $a_n \neq 0$, the **degree** is $\deg(f) = n$, and a_n is the **leading coefficient**. If $a_n = 1$, we call f **monic**. The zero polynomial has degree $-\infty$ by convention.

> **Definition**
>
> **Addition** and **multiplication** in $R[x]$ are defined by:
>
> $$\left(\sum_{i=0}^{n} a_i x^i\right) + \left(\sum_{i=0}^{m} b_i x^i\right) = \sum_{i=0}^{\max(n,m)} (a_i + b_i) x^i$$
>
> $$\left(\sum_{i=0}^{n} a_i x^i\right) \cdot \left(\sum_{j=0}^{m} b_j x^j\right) = \sum_{k=0}^{n+m} \left(\sum_{i+j=k} a_i b_j\right) x^k$$

Theorem

If R is a commutative ring with unity, then $R[x]$ is also a commutative ring with unity, where the additive identity is 0 and the multiplicative identity is 1.

Proof. Ring axioms follow directly from those of R. Commutativity of multiplication: $(a_i x^i)(b_j x^j) = (a_i b_j)x^{i+j} = (b_j a_i)x^{j+i} = (b_j x^j)(a_i x^i)$. Associativity and distributivity follow similarly. The unity is the constant polynomial $1 \in R$. ∎

Proposition

If R is an integral domain, then for non-zero polynomials $f, g \in R[x]$:

$$\deg(fg) = \deg(f) + \deg(g)$$

In particular, $R[x]$ is also an integral domain.

Proof. If $f(x) = a_n x^n + \cdots$ and $g(x) = b_m x^m + \cdots$ with $a_n, b_m \neq 0$, then the leading coefficient of fg is $a_n b_m$. Since R is an integral domain, $a_n b_m \neq 0$, so $\deg(fg) = n + m$. Thus $fg \neq 0$, making $R[x]$ an integral domain. ∎

Theorem

(Universal Property) Let R be a commutative ring with unity, S a commutative ring containing R, and $s \in S$. There exists a unique ring homomorphism $\varphi : R[x] \to S$ such that $\varphi|_R = \mathrm{id}_R$ and $\varphi(x) = s$.

Proof. Define $\varphi(f(x)) = f(s)$, the evaluation at s. This is the unique homomorphism extending the identity on R and sending $x \mapsto s$. Well-definedness and homomorphism properties follow from the ring structure. ∎

Example

$\mathbb{Z}[x]$**:** Polynomials with integer coefficients. Examples: $2x^3 - 5x + 7$, $x^{100} + 1$. Not every polynomial has roots in $\mathbb{Z}$, e.g., $x^2 - 2$.

Example

$\mathbb{Q}[x]$**:** Polynomials with rational coefficients. The polynomial $x^2 - 2$ has no roots in $\mathbb{Q}$, but can be factored in $\mathbb{R}[x]$ as $(x - \sqrt{2})(x + \sqrt{2})$.

Example

$\mathbb{Z}_p[x]$ for prime p: Polynomials over the finite field $\mathbb{Z}_p$. For instance, in $\mathbb{Z}_2[x]$, we have $x^2 + x + 1$ which is irreducible.

Example

Multiple indeterminates: $R[x, y] = (R[x])[y]$ is the ring of polynomials in two variables. For instance, $x^2y + 3xy^2 - 7 \in \mathbb{Z}[x, y]$.

10.2 Division Algorithm

Just as integers possess a division algorithm yielding quotient and remainder, polynomials over a field enjoy an analogous property.

Theorem

(Division Algorithm for Polynomials) Let F be a field and $f(x), g(x) \in F[x]$ with $g(x) \neq 0$. Then there exist unique polynomials $q(x), r(x) \in F[x]$ such that

$$f(x) = q(x)g(x) + r(x)$$

where either $r(x) = 0$ or $\deg(r) < \deg(g)$.

Proof. **Existence:** If $f = 0$ or $\deg(f) < \deg(g)$, take $q = 0$ and $r = f$. Otherwise, proceed by strong induction on $\deg(f) = n \geq \deg(g) = m$. Write $f(x) = a_n x^n + \cdots$ and $g(x) = b_m x^m + \cdots$ with $a_n, b_m \neq 0$. Consider

$$f_1(x) = f(x) - \frac{a_n}{b_m} x^{n-m} g(x)$$

The leading term $a_n x^n$ cancels, so $\deg(f_1) < n$. By induction, $f_1(x) = q_1(x)g(x) + r(x)$ with $\deg(r) < \deg(g)$ or $r = 0$. Thus

$$f(x) = \left(q_1(x) + \frac{a_n}{b_m} x^{n-m}\right) g(x) + r(x)$$

Set $q(x) = q_1(x) + \frac{a_n}{b_m} x^{n-m}$.

Uniqueness: Suppose $f = qg + r = q'g + r'$ with $\deg(r), \deg(r') < \deg(g)$. Then $(q - q')g = r' - r$. If $q \neq q'$, then $\deg((q - q')g) = \deg(q - q') + \deg(g) \geq \deg(g)$, but $\deg(r' - r) < \deg(g)$, a contradiction. Thus $q = q'$ and $r = r'$. ∎

Example

Divide $f(x) = 2x^4 + 3x^3 - x + 5$ by $g(x) = x^2 + 1$ in $\mathbb{Q}[x]$.

Using polynomial long division:

	$2x^2 + 3x - 2$
$x^2 + 1$	$2x^4 + 3x^3 + 0x^2 - x + 5$
	$2x^4 + 0x^3 + 2x^2$
	$3x^3 - 2x^2 - x$
	$3x^3 + 0x^2 + 3x$
	$-2x^2 - 4x + 5$
	$-2x^2 + 0x - 2$
	$-4x + 7$

Thus $q(x) = 2x^2 + 3x - 2$ and $r(x) = -4x + 7$.

Definition

The **Euclidean algorithm** for polynomials computes the greatest common divisor. For $f, g \in F[x]$ with $g \neq 0$:

1. Apply division algorithm: $f = qg + r$ with $\deg(r) < \deg(g)$
2. If $r = 0$, then $\gcd(f, g) = g$
3. Otherwise, $\gcd(f, g) = \gcd(g, r)$ and repeat

Example

Find $\gcd(x^4 + x^3 + x^2 + x + 1, x^3 - 1)$ in $\mathbb{Q}[x]$.

$x^4 + x^3 + x^2 + x + 1 = (x)(x^3 - 1) + (x^3 + x^2 + 2x + 1)$
$x^3 - 1 = (1)(x^3 + x^2 + 2x + 1) + (-x^2 - 2x - 2)$
$x^3 + x^2 + 2x + 1 = (-x)(-x^2 - 2x - 2) + (-x + 1)$
$-x^2 - 2x - 2 = (x + 1)(-x + 1) + (-1)$
$-x + 1 = (x - 1)(-1) + 0$

Thus $\gcd(f, g) = -1$, or normalizing, $\gcd(f, g) = 1$ (coprime).

10.3 Ideals in F[x]

Theorem

($F[x]$ is a Principal Ideal Domain) Let F be a field. Every ideal I in $F[x]$ is principal, i.e., $I = \langle f(x) \rangle$ for some $f(x) \in F[x]$.

Proof. If $I = \{0\}$, then $I = \langle 0 \rangle$. Otherwise, let $f(x) \in I$ be a non-zero polynomial of minimal degree. Clearly $\langle f \rangle \subseteq I$. Conversely, let $g(x) \in I$. By the division algorithm, $g(x) = q(x)f(x) + r(x)$ with $\deg(r) < \deg(f)$ or $r = 0$. Since $g, f \in I$, we have $r = g - qf \in I$. By minimality of $\deg(f)$, we must have $r = 0$. Thus $g \in \langle f \rangle$, so $I = \langle f \rangle$. ∎

Definition

For $f, g \in F[x]$, the **greatest common divisor** $\gcd(f, g)$ is the monic polynomial of highest degree dividing both f and g. The **least common multiple** $\operatorname{lcm}(f, g)$ is the monic polynomial of least degree divisible by both f and g.

Proposition

For $f, g \in F[x]$, the ideal $\langle f, g \rangle$ is principal, generated by $d = \gcd(f, g)$. Moreover, there exist $s, t \in F[x]$ such that

$$d = sf + tg$$

(Bézout's identity for polynomials).

Proof. Since $F[x]$ is a PID, $\langle f, g \rangle = \langle d \rangle$ for some $d \in F[x]$. Clearly $d | f$ and $d | g$. If $h | f$ and $h | g$, then $h | d$ since $d \in \langle f, g \rangle$. Thus $d = \gcd(f, g)$. Since $d \in \langle f, g \rangle$, we have $d = sf + tg$ for some $s, t \in F[x]$. ∎

Example

Find $s, t \in \mathbb{Q}[x]$ such that $1 = s(x^2 - 1) + t(x^2 + 2x + 1)$.

Note $x^2 + 2x + 1 = (x + 1)^2$ and $x^2 - 1 = (x - 1)(x + 1)$, so gcd $= x + 1$. Using extended Euclidean algorithm:
$x^2 + 2x + 1 = (1)(x^2 - 1) + (3x + 2)$
$x^2 - 1 = \frac{x-2}{3}(3x + 2) + \frac{1}{3}$

Back-substituting and normalizing:
$1 = 3(x^2 - 1) - (x - 2)(3x + 2) = 3(x^2 - 1) - (x - 2)((x^2 + 2x + 1) - (x^2 - 1))$
$= 3(x^2 - 1) - (x - 2)(x^2 + 2x + 1) + (x - 2)(x^2 - 1)$
$= (x + 1)(x^2 - 1) - (x - 2)(x^2 + 2x + 1)$

Thus $s(x) = x + 1$ and $t(x) = -(x - 2) = -x + 2$.

10.4 Irreducibility

Definition

A non-constant polynomial $f(x) \in F[x]$ is **irreducible** over F if it cannot be factored as $f = gh$ with $g, h \in F[x]$ both non-constant. Otherwise, f is **reducible**.

Irreducibility depends on the field. For example, $x^2 - 2$ is irreducible over $\mathbb{Q}$ but factors as $(x - \sqrt{2})(x + \sqrt{2})$ over $\mathbb{R}$.

Proposition

Over $\mathbb{C}$, the only irreducible polynomials are linear (degree 1). Over $\mathbb{R}$, irreducibles are linear or quadratic with no real roots.

Proof. By the Fundamental Theorem of Algebra, every polynomial in $\mathbb{C}[x]$ factors completely into linear factors. For $\mathbb{R}[x]$, complex roots come in conjugate pairs, giving real quadratic factors $x^2 + bx + c$ with $b^2 - 4c < 0$. ∎

Theorem

(Eisenstein's Criterion) Let $f(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$. If there exists a prime p such that:

1. $p \nmid a_n$ (p does not divide leading coefficient)
2. $p \mid a_i$ for all $i < n$ (p divides all other coefficients)
3. $p^2 \nmid a_0$ (p² does not divide constant term)

then $f(x)$ is irreducible over $\mathbb{Q}$.

Proof. Suppose $f = gh$ with $g, h \in \mathbb{Z}[x]$ non-constant (clearing denominators from $\mathbb{Q}[x]$ if necessary). Write $g(x) = b_r x^r + \cdots + b_0$ and $h(x) = c_s x^s + \cdots + c_0$. Then $a_0 = b_0 c_0$. Since $p \mid a_0$ but $p^2 \nmid a_0$, exactly one of b_0, c_0 is divisible by p, say $p \mid b_0$ and $p \nmid c_0$.

Since $p \nmid a_n = b_r c_s$, we have $p \nmid b_r$ (as p is prime). Let k be the smallest index with $p \nmid b_k$. Then $k \geq 1$ (since $p \mid b_0$). Consider $a_k = b_0 c_k + b_1 c_{k-1} + \cdots + b_k c_0$. By hypothesis, $p \mid a_k$. Each term $b_i c_{k-i}$ with $i < k$ is divisible by p, but $b_k c_0$ is not divisible by p (since $p \nmid b_k$ and $p \nmid c_0$). This contradicts $p \mid a_k$. Hence f is irreducible. ∎

Example

Show $f(x) = x^4 + 2x^3 + 6x^2 + 4x + 10$ is irreducible over $\mathbb{Q}$.

Apply Eisenstein with $p = 2$: $2 \nmid 1$ (leading coefficient), $2 \mid 2, 6, 4, 10$ (other coefficients), and $4 \nmid 10$ (constant term). Thus f is irreducible over $\mathbb{Q}$.

Example

The cyclotomic polynomial $\Phi_p(x) = 1 + x + x^2 + \cdots + x^{p-1}$ is irreducible over $\mathbb{Q}$ for prime p.

Consider $\Phi_p(x+1) = \frac{(x+1)^p - 1}{x} = \sum_{k=0}^{p-1} \binom{p}{k+1} x^k$. For $0 < k < p$, $p \mid \binom{p}{k+1}$, and $\binom{p}{1} = p$ is not divisible by p^2. Eisenstein applies with prime p, so $\Phi_p(x+1)$ is irreducible, hence $\Phi_p(x)$ is irreducible.

Proposition

(Mod p Irreducibility Test) Let $f(x) \in \mathbb{Z}[x]$ be primitive (gcd of coefficients is 1) with degree n. If $\bar{f}(x) \in \mathbb{Z}_p[x]$ (reduction mod p) is irreducible and $\deg(\bar{f}) = n$, then f is irreducible over $\mathbb{Q}$.

Example

Show $f(x) = x^3 + x + 1$ is irreducible over $\mathbb{Q}$.

In $\mathbb{Z}_2[x]$, $\bar{f}(x) = x^3 + x + 1$. Check: $\bar{f}(0) = 1$, $\bar{f}(1) = 1 + 1 + 1 = 1$. Since $\bar{f}$ has no roots in $\mathbb{Z}_2$ and degree 3, it's irreducible. Thus f is irreducible over $\mathbb{Q}$.

10.5 Unique Factorization

Theorem

($F[x]$ is a Unique Factorization Domain) Let F be a field. Every non-constant polynomial $f(x) \in F[x]$ can be written uniquely (up to order and units) as

$$f(x) = c \cdot p_1(x)^{e_1} p_2(x)^{e_2} \cdots p_k(x)^{e_k}$$

where $c \in F^\times$, the p_i are distinct monic irreducible polynomials, and $e_i > 0$.

Proof. **Existence:** Induction on degree. If $\deg(f) = 1$, then f is irreducible. Otherwise, if f is irreducible, done. If $f = gh$ with $\deg(g), \deg(h) < \deg(f)$, apply induction to g and h.

Uniqueness: Suppose $f = c\prod p_i^{e_i} = d\prod q_j^{f_j}$ with monic irreducibles p_i, q_j. Then $p_1 | \prod q_j^{f_j}$. Since p_1 is irreducible (hence prime in the PID $F[x]$), $p_1 | q_j$ for some j. Both are monic irreducibles, so $p_1 = q_j$. Cancel and repeat by induction on degree. ∎

Definition

A polynomial $f(x) \in \mathbb{Z}[x]$ is **primitive** if the gcd of its coefficients is 1.

(Gauss's Lemma) The product of primitive polynomials in $\mathbb{Z}[x]$ is primitive.

Proof. Let $f, g \in \mathbb{Z}[x]$ be primitive. Suppose fg is not primitive, so some prime p divides all coefficients of fg. Consider $\bar{f}, \bar{g} \in \mathbb{Z}_p[x]$, the reductions mod p. Since p does not divide all coefficients of f or g (primitivity), $\bar{f} \neq 0$ and $\bar{g} \neq 0$. But $\overline{fg} = \bar{f} \cdot \bar{g}$, and $\mathbb{Z}_p[x]$ is an integral domain (as $\mathbb{Z}_p$ is a field), so $\bar{f} \cdot \bar{g} \neq 0$. This contradicts $p \mid fg$. Thus fg is primitive. ∎

Theorem

A primitive polynomial $f(x) \in \mathbb{Z}[x]$ is irreducible over $\mathbb{Q}$ if and only if it's irreducible over $\mathbb{Z}$.

Proof. If $f = gh$ in $\mathbb{Q}[x]$ with g, h non-constant, clear denominators: $cf = \tilde{g}\tilde{h}$ with $\tilde{g}, \tilde{h} \in \mathbb{Z}[x]$ primitive (by Gauss). Thus $c = \text{cont}(f) = 1$, so $f = \tilde{g}\tilde{h}$ in $\mathbb{Z}[x]$. Conversely, factorization in $\mathbb{Z}[x]$ gives one in $\mathbb{Q}[x]$. ∎

Example

Factor $f(x) = x^4 - 5x^2 + 6$ over $\mathbb{Q}$.

Substitute $y = x^2$: $y^2 - 5y + 6 = (y-2)(y-3) = (x^2-2)(x^2-3)$.

Over $\mathbb{R}$: $(x-\sqrt{2})(x+\sqrt{2})(x-\sqrt{3})(x+\sqrt{3})$.

10.6 Roots and Factorization

Theorem

(Factor Theorem) Let F be a field and $f(x) \in F[x]$. Then $a \in F$ is a root of f (i.e., $f(a) = 0$) if and only if $(x-a) | f(x)$.

Proof. By division algorithm, $f(x) = (x-a)q(x) + r$ with $\deg(r) < 1$, so $r \in F$. Evaluating at $x = a$: $f(a) = 0 \cdot q(a) + r = r$. Thus $f(a) = 0$ if and only if $r = 0$, i.e., $(x-a) | f$. ∎

(Remainder Theorem) The remainder when $f(x)$ is divided by $(x-a)$ is $f(a)$.

Definition

If $(x-a)^m | f(x)$ but $(x-a)^{m+1} \nmid f(x)$, then a is a root of **multiplicity** m.

Proposition

A polynomial of degree n over a field F has at most n roots in F (counting multiplicities).

Proof. Induction on n. If f has a root a, then $f(x) = (x-a)g(x)$ with $\deg(g) = n-1$. By induction, g has at most $n-1$ roots, so f has at most n roots. ∎

Theorem

(Rational Root Theorem) Let $f(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$ with $a_n \neq 0$ and $a_0 \neq 0$. If $\frac{p}{q}$ is a rational root in lowest terms, then $p|a_0$ and $q|a_n$.

Proof. We have $f(p/q) = 0$, so $a_n(p/q)^n + \cdots + a_0 = 0$. Multiply by q^n:

$$a_n p^n + a_{n-1} p^{n-1} q + \cdots + a_0 q^n = 0$$

Thus $a_n p^n = -q(a_{n-1}p^{n-1} + \cdots + a_0 q^{n-1})$, so $q|a_n p^n$. Since $\gcd(p, q) = 1$, we have $q|a_n$.
Similarly, $a_0 q^n = -p(a_n p^{n-1} + \cdots + a_1 q^{n-1})$, so $p|a_0 q^n$, hence $p|a_0$. ∎

Example

Find all rational roots of $f(x) = 2x^3 - 3x^2 - 11x + 6$.

Possible rational roots: $\frac{p}{q}$ where $p|6$ and $q|2$. Thus $p \in \{\pm 1, \pm 2, \pm 3, \pm 6\}$ and $q \in \{\pm 1, \pm 2\}$.
Candidates: $\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}$.

Testing: $f(1) = 2 - 3 - 11 + 6 = -6$, $f(-1) = -2 - 3 + 11 + 6 = 12$,
$f(2) = 16 - 12 - 22 + 6 = -12$, $f(3) = 54 - 27 - 33 + 6 = 0$ ✓
$f(\frac{1}{2}) = \frac{1}{4} - \frac{3}{4} - \frac{11}{2} + 6 = 0$ ✓

By synthetic division: $f(x) = (x - 3)(2x^2 + 3x - 2) = (x - 3)(2x - 1)(x + 2)$.
Roots: $x = 3, \frac{1}{2}, -2$.

Example

Show $f(x) = x^3 - 2x + 4$ has no rational roots.

Possible rational roots: $\pm 1, \pm 2, \pm 4$. Testing: $f(1) = 3$, $f(-1) = 3$, $f(2) = 8$, $f(-2) = -8$, $f(4) = 60$, $f(-4) = -60$. None equal zero, so f has no rational roots. Since f is cubic, it must be irreducible over $\mathbb{Q}$.

Greek Meditation

Polynomials as Cosmic Harmonies: The Pythagorean Music of the Spheres

The Pythagoreans believed that number was the arche—the fundamental principle

underlying all reality. In polynomial rings, we witness this vision realized: the indeterminate x becomes the cosmic string whose vibrations encode all possible numerical relationships. Each polynomial is a harmonic sequence, its roots the resonant frequencies where the cosmic music touches silence.

Just as Pythagoras discovered that musical intervals correspond to simple ratios—octave $2:1$, fifth $3:2$, fourth $4:3$—so do polynomial factorizations reveal the hidden rationality beneath algebraic complexity. The irreducible polynomials are the primes of this harmonic universe, the fundamental notes that cannot be decomposed further, from which all melodies are composed.

Plato's cave of shadows finds its echo in field extensions: the polynomial $x^2 - 2$ is irreducible over $\mathbb{Q}$, a mathematical shadow on the wall. But in $\mathbb{R}$, we see its true form: $(x - \sqrt{2})(x + \sqrt{2})$. Each extension of the field is an ascent from the cave, revealing deeper truths hidden from the prisoners below.

Eisenstein's criterion, that sublime test of irreducibility, embodies the Greek ideal of *logos*—rational order emerging from prime structure. That a single prime can certify irreducibility speaks to the interconnectedness of all mathematical truth, the cosmic web where number, form, and logic interweave.

Zen Meditation

Roots and Emptiness: Where the Polynomial Touches the Void

A polynomial without roots is like a koan without answer—complete in its incompleteness, perfect in its irreducibility. The equation $x^2 + 1 = 0$ has no solution in the real numbers, yet this absence is not deficiency but invitation. The void calls forth new realms of being.

Consider the zero polynomial, the ultimate emptiness: no degree, no coefficients, only pure negativity (degree $-\infty$). Yet this nothingness generates the zero ideal, contains all structure within its formless form. Like the Buddhist śūnyatā, mathematical zero is a profound, potential filled, awareness driven operator current manifesting-unmanifesting form-void, the womb of all possibility.

The Factor Theorem teaches: to know a root is to divide, to separate unity into parts. $(x - a) | f(x)$ precisely when $f(a) = 0$—where the polynomial meets the void, there division becomes possible. Enlightenment comes at the root, the place of dissolution.

Unique factorization is the doctrine of dependent origination: every polynomial arises from causes and conditions, the irreducibles. No polynomial has independent existence; each is empty of inherent nature, reducible to primes. Yet the primes themselves are irreducible,

> emptiness as ultimate reality.
>
> In Zen archery, one aims without aiming, shoots without shooter. In polynomial algebra, we manipulate formal symbols—x means nothing yet contains all meaning. The indeterminate is indeterminate, free from the bondage of particular values. This is the freedom of pure form, mathematics as meditation on emptiness.

Practice Problems

1. Prove that if R is an integral domain, then the units of $R[x]$ are precisely the units of R (constant polynomials).

   > ***Solution.*** Let $u(x) \in R[x]$ be a unit, so $u(x)v(x) = 1$ for some $v(x) \in R[x]$. Then $\deg(u) + \deg(v) = \deg(1) = 0$. Since degrees are non-negative (in an integral domain), $\deg(u) = \deg(v) = 0$. Thus $u, v \in R$ are constant polynomials, and they are units in R.

2. Divide $f(x) = x^5 + 2x^3 - x + 7$ by $g(x) = x^2 - 2x + 1$ in $\mathbb{Q}[x]$ to find quotient and remainder.

   > ***Solution.*** By polynomial long division:
   >
   > $$x^5 + 2x^3 - x + 7 = (x^3 + 2x^2 + 5x + 9)(x^2 - 2x + 1) + (16x - 2)$$
   >
   > Thus $q(x) = x^3 + 2x^2 + 5x + 9$ and $r(x) = 16x - 2$.

3. Compute $\gcd(x^4 - 1, x^3 + x^2 + x + 1)$ in $\mathbb{Q}[x]$ using the Euclidean algorithm.

   > ***Solution.*** $x^4 - 1 = (x)(x^3 + x^2 + x + 1) + (-x^2 - x - 1)$
   > $x^3 + x^2 + x + 1 = (-x + 1)(-x^2 - x - 1) + 0$
   >
   > Thus $\gcd = -x^2 - x - 1$, or normalizing, $\gcd = x^2 + x + 1$.

4. Prove that in a principal ideal domain, every irreducible element is prime.

   > ***Solution.*** Let p be irreducible and suppose $p|ab$. Consider $I = \langle p, a \rangle$. Since the ring is a PID, $I = \langle d \rangle$ for some d. Then $d|p$ and $d|a$. Since p is irreducible, either $d \sim p$ or d is a unit. If $d \sim p$, then $p|a$. If d is a unit, then $1 \in I$, so $1 = sp + ta$ for some s, t. Multiplying by b: $b = spb + tab$. Since $p|ab$, we have $p|b$. Thus $p|a$ or $p|b$, so p is prime.

5. Show that x^2+1 is irreducible over $\mathbb{Z}_2[x]$ but reducible over $\mathbb{Z}_3[x]$.

> ***Solution.*** In $\mathbb{Z}_2[x]$: Check roots: $0^2+1=1\neq 0$, $1^2+1=0\neq 0$ (in $\mathbb{Z}_2$, $1+1=0$ but we need to check: $1+1\equiv 2\equiv 0$, so actually $1^2+1=1+1=0$). Thus $x=1$ is a root, so $x^2+1=(x+1)^2$ in $\mathbb{Z}_2[x]$, actually reducible.
>
> Let me reconsider: In $\mathbb{Z}_3[x]$: $0^2+1=1$, $1^2+1=2$, $2^2+1=4+1=5\equiv 2 \pmod 3$. No roots, but degree 2, so check factorization: would need $(x+a)(x+b)=x^2+(a+b)x+ab$. Need $a+b=0$ and $ab=1$ in $\mathbb{Z}_3$. From $ab=1$: $(a,b)\in\{(1,1),(2,2)\}$. But $1+1=2\neq 0$ and $2+2=4\equiv 1\neq 0$. So irreducible over $\mathbb{Z}_3$.
>
> For $\mathbb{Z}_2$: Actually $x^2+1=(x+1)^2=x^2+2x+1=x^2+1$ (since $2x=0$ in $\mathbb{Z}_2$), so it's irreducible if it has no linear factors, i.e., no roots. Check: $0^2+1=1\neq 0$, $1^2+1=0$. So $x=1$ is a root, meaning $x^2+1=(x-1)q=(x+1)q$ in $\mathbb{Z}_2[x]$. Thus $x^2+1=(x+1)^2$, reducible.
>
> Correction: Over $\mathbb{Z}_5$, we have $2^2=4\equiv -1$, so $x^2+1=(x+2)(x+3)=(x-2)(x-3)$ (using $-2\equiv 3$ and $-3\equiv 2$).
>
> Proper answer: Over $\mathbb{Z}_2$: $x^2+1=(x+1)^2$, reducible. Over $\mathbb{Z}_3$: no roots and no factorization, irreducible.

6. Use Eisenstein's criterion to show that $f(x)=x^5-3x^3+15x-21$ is irreducible over $\mathbb{Q}$.

> ***Solution.*** Try $p=3$: $3\nmid 1$ (leading coeff), $3|0$ (coeff of x^4), $3|-3$, $3|0$, $3|15$, $3|-21$ (all lower coefficients), and $9\nmid -21$ (constant term). Eisenstein applies with $p=3$, so f is irreducible over $\mathbb{Q}$.

7. Prove that $\Phi_5(x)=x^4+x^3+x^2+x+1$ is irreducible over $\mathbb{Q}$.

> ***Solution.*** Consider $\Phi_5(x+1)=\frac{(x+1)^5-1}{x}=\frac{x^5+5x^4+10x^3+10x^2+5x+1-1}{x}=x^4+5x^3+10x^2+10x+5$. Apply Eisenstein with $p=5$: $5\nmid 1$, $5|5,10,10,5$, and $25\nmid 5$. Thus $\Phi_5(x+1)$ is irreducible, hence $\Phi_5(x)$ is irreducible over $\mathbb{Q}$.

8. If $f(x),g(x)\in F[x]$ have no common factors, show that the ideal $\langle f,g\rangle=F[x]$.

> ***Solution.*** Since $\gcd(f,g)=1$ and $F[x]$ is a PID, the ideal $\langle f,g\rangle=\langle\gcd(f,g)\rangle=\langle 1\rangle=F[x]$.

9. Factor $f(x)=x^4-2x^2-3$ completely over $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$.

> ***Solution.*** Set $y = x^2$: $y^2 - 2y - 3 = (y-3)(y+1) = (x^2-3)(x^2+1)$.
>
> **Over $\mathbb{Q}$:** $f(x) = (x^2-3)(x^2+1)$ (both irreducible over $\mathbb{Q}$).
> **Over $\mathbb{R}$:** $f(x) = (x-\sqrt{3})(x+\sqrt{3})(x^2+1)$.
> **Over $\mathbb{C}$:** $f(x) = (x-\sqrt{3})(x+\sqrt{3})(x-i)(x+i)$.

10. Show that if $f(x) \in \mathbb{Z}[x]$ is monic and has a root $a \in \mathbb{Q}$, then $a \in \mathbb{Z}$.

> ***Solution.*** By Rational Root Theorem, $a = \frac{p}{q}$ in lowest terms with $p | a_0$ and $q | a_n = 1$ (since f is monic). Thus $q = \pm 1$, so $a \in \mathbb{Z}$.

11. Find all roots (with multiplicities) of $f(x) = x^4 - 4x^3 + 6x^2 - 4x + 1$ over $\mathbb{C}$.

> ***Solution.*** Recognize $f(x) = (x-1)^4$ (binomial expansion). Thus the only root is $x = 1$ with multiplicity 4.

12. Prove Gauss's Lemma: the product of primitive polynomials in $\mathbb{Z}[x]$ is primitive.

> ***Solution.*** (See proof in Section 10.5)

13. Show that $x^2 + y^2 - 1$ is irreducible in $\mathbb{R}[x, y]$.

> ***Solution.*** View as polynomial in $\mathbb{R}[y][x]$: $x^2 + (y^2 - 1)$. This is degree 2 in x, and would factor only if it has a root in $\mathbb{R}[y]$, i.e., $x = g(y)$ with $g(y)^2 = 1 - y^2$. But $1 - y^2$ is not a square in $\mathbb{R}[y]$ (consider degrees and leading coefficients). Alternatively, the zero set $x^2 + y^2 = 1$ is the unit circle, which is an irreducible variety, hence the polynomial is irreducible.

14. If $F \subseteq K$ are fields and $f(x) \in F[x]$ is irreducible, show that any irreducible factor of f in $K[x]$ has degree dividing $\deg(f)$.

> ***Solution.*** Let $g(x) \in K[x]$ be irreducible with $g | f$. Then $f = gh$ for some $h \in K[x]$. Taking degrees: $\deg(f) = \deg(g) + \deg(h)$, so $\deg(g) | \deg(f)$.

15. Determine whether $f(x) = x^4 + 1$ is irreducible over $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{Z}_2$.

> ***Solution.*** **Over $\mathbb{Q}$:** Try Eisenstein—fails for all primes. Instead, factor over $\mathbb{C}$: roots are $e^{i\pi/4}, e^{3i\pi/4}, e^{5i\pi/4}, e^{7i\pi/4}$, giving
>
> $$x^4 + 1 = (x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2}x + 1)$$
>
> over $\mathbb{R}$. Check if these factors are in $\mathbb{Q}[x]$—no, since coefficients involve $\sqrt{2}$. Actually, $x^4 + 1$ is irreducible over $\mathbb{Q}$ (can be shown using $x^4 + 1 = \Phi_8(x)$, the 8th cyclotomic polynomial).

Over $\mathbb{R}$: $x^4+1=(x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)$, reducible.
Over $\mathbb{Z}_2$: $x^4+1=x^4+1$. Check roots: $0^4+1=1$, $1^4+1=0$. So $x=1$ is a root, thus $(x+1)|(x^4+1)$. Division gives $x^4+1=(x+1)(x^3+x^2+x+1)$. Check if x^3+x^2+x+1 factors: $1^3+1^2+1+1=4=0$ in $\mathbb{Z}_2$, so $x+1$ divides again. Thus $x^4+1=(x+1)^4$ in $\mathbb{Z}_2[x]$, reducible.

16. Find all monic irreducible polynomials of degree 2 over $\mathbb{Z}_3$.

Solution. Degree 2 monic: x^2+bx+c with $b,c\in\mathbb{Z}_3=\{0,1,2\}$. Irreducible iff no roots.

x^2: reducible ($x\cdot x$).
x^2+1: roots? $0^2+1=1$, $1^2+1=2$, $2^2+1=5\equiv 2$. No roots, irreducible. ✓
x^2+2: $0^2+2=2$, $1^2+2=0$. Root at $x=1$, reducible.
x^2+x: $0^2+0=0$. Root at $x=0$, reducible.
x^2+x+1: $0^2+0+1=1$, $1^2+1+1=0$. Root at $x=1$, reducible.
x^2+x+2: $0^2+0+2=2$, $1^2+1+2=4=1$, $2^2+2+2=8=2$. No roots, irreducible. ✓
x^2+2x: $0^2+0=0$. Root at $x=0$, reducible.
x^2+2x+1: $1^2+2+1=4=1$, $2^2+4+1=9=0$. Root at $x=2$, reducible.
x^2+2x+2: $0^2+0+2=2$, $1^2+2+2=5=2$, $2^2+4+2=10=1$. No roots, irreducible. ✓

Thus: x^2+1, x^2+x+2, x^2+2x+2.

17. Show that $f(x)=x^n-a$ is irreducible over $\mathbb{Q}$ if a is not an n-th power of a rational number and n is prime.

Solution. Let $\omega=\sqrt[n]{a}$ be a root in $\mathbb{C}$. Then $f(x)=\prod_{j=0}^{n-1}(x-\omega\zeta^j)$ where $\zeta=e^{2\pi i/n}$. If f factors over $\mathbb{Q}$ as $f=gh$ non-trivially, then each factor corresponds to a subset of roots. Since n is prime, the Galois group acts transitively, so any proper factor would require intermediate degree, contradicting irreducibility. More rigorously: if g has a root $\omega\zeta^j$, then $[\mathbb{Q}(\omega\zeta^j):\mathbb{Q}]=\deg(g)<n$. But $[\mathbb{Q}(\omega):\mathbb{Q}]=n$ (since a is not an n-th power), contradiction unless f is irreducible.

Alternatively, use Eisenstein: if $a=p^kb$ with p prime, $p\nmid b$, and k not divisible by n, then x^n-a is irreducible by Eisenstein on x^n-p^kb after suitable transformation.

18. Let $f(x)=x^3+ax+b\in\mathbb{Q}[x]$. Show that f has a multiple root if and only if $4a^3+27b^2=0$ (the discriminant condition).

Solution. f has multiple root α iff $f(\alpha)=f'(\alpha)=0$. We have $f'(x)=3x^2+a$. If α is a common root, then $\alpha^2=-a/3$ and $\alpha^3+a\alpha+b=0$. Substituting:

$\alpha \cdot \alpha^2 + a\alpha + b = -a\alpha/3 + a\alpha + b = 2a\alpha/3 + b = 0$, so $\alpha = -3b/(2a)$ (if $a \neq 0$). Then $\alpha^2 = 9b^2/(4a^2) = -a/3$, giving $27b^2 = -4a^3$, i.e., $4a^3 + 27b^2 = 0$. If $a = 0$, then $f(x) = x^3 + b$, which has simple roots unless $b = 0$. The discriminant is $-27b^2$, which is zero iff $b = 0$. Thus the condition $4a^3 + 27b^2 = 0$ characterizes multiple roots.

19. Prove that $\mathbb{Q}[x]$ is not a Euclidean domain.

Solution. Actually, $\mathbb{Q}[x]$ *is* a Euclidean domain with the degree function as Euclidean norm (since $\mathbb{Q}$ is a field). The statement in the problem is incorrect. However, $\mathbb{Z}[x]$ is *not* a Euclidean domain (though it is a UFD).

Corrected problem: $\mathbb{Z}[x]$ is not a Euclidean domain. Proof sketch: If $\mathbb{Z}[x]$ were Euclidean with norm ν, consider 2 and x. We'd need $\nu(2) < \nu(x)$ or vice versa. But $2 \nmid x$ and $x \nmid 2$, and there's no "remainder" in the Euclidean sense. More precisely, the ideal $\langle 2, x \rangle$ is not principal (it equals $\{2f(x) + xg(x) : f, g \in \mathbb{Z}[x]\}$, which contains no generator), contradicting the Euclidean domain property.

20. Let $f(x) = x^4 + ax^2 + b \in \mathbb{Q}[x]$. Find conditions on a, b for f to factor over $\mathbb{Q}$ as a product of two quadratics.

Solution. Suppose $f(x) = (x^2 + px + q)(x^2 + rx + s)$ with $p, q, r, s \in \mathbb{Q}$. Expanding:

$$x^4 + (p + r)x^3 + (q + s + pr)x^2 + (ps + qr)x + qs$$

Matching coefficients:

1. $p + r = 0 \Rightarrow r = -p$
2. $q + s + pr = a \Rightarrow q + s - p^2 = a$
3. $ps + qr = 0 \Rightarrow ps - qp = 0 \Rightarrow p(s - q) = 0$
4. $qs = b$

Case 1: $p = 0$. Then $q + s = a$ and $qs = b$, so q, s are roots of $t^2 - at + b = 0$. This has rational solutions iff $a^2 - 4b$ is a perfect square.

Case 2: $s = q$. Then $2q - p^2 = a$ and $q^2 = b$, so $q = \pm\sqrt{b}$ and $p^2 = 2q - a = \pm 2\sqrt{b} - a$. For rational p, need $\pm 2\sqrt{b} - a$ to be a perfect square in $\mathbb{Q}$, requiring b to be a perfect square.

Condition: Either (i) $a^2 - 4b$ is a perfect square, or (ii) b is a perfect square and $2\sqrt{b} - a$ or $-2\sqrt{b} - a$ is a perfect square.

Volume 3: Self-Contained Worlds

Ὁ κόσμος τάξις ἐστί

The cosmos is order—

and each field is a cosmos complete:

bounded yet infinite,

contained yet containing all possibility.

In the heaven of Indra, Buddhist cosmology places an infinite net. At each node gleams a perfect jewel, and in each jewel reflects all others—infinite mirrors containing infinite reflections, the whole present in every part. This is how fields relate: each complete unto itself, yet each reflecting and containing principles that manifest in all others.

A field is mathematical completion. Addition works without exception. Multiplication works without exception (save for zero). Every equation has solutions or proves itself impossible—there are no unanswered questions left dangling. Division always succeeds. Subtraction never fails. Within a field's boundaries, algebra achieves closure, finality, a kind of perfection.

On Completion and Containment:

The rational numbers form a field—the first field, the template. Every fraction exists: 1/2, 355/113, the ratio of any integer to any non-zero integer. This seems complete until we ask for √2. The

rationals cannot provide it. They are complete *as a field*, yet incomplete *as a universe of number*.

So we extend: adjoin √2 to the rationals, and suddenly a larger field appears—$\mathbb{Q}(\sqrt{2})$, all numbers of the form $a + b\sqrt{2}$ where a and b are rational. This new field contains the old completely, adds new elements, yet maintains perfect field structure. We can extend further: adjoin $\sqrt[3]{2}$, or i, or π, building larger and larger fields like Chinese boxes, each nested inside the next, each complete in itself.

Here's what startles: the field doesn't just "contain" these new elements. It *dreams them into necessity*. The rationals contain the polynomial $x^2 - 2$. This polynomial, having no rational roots, is a kind of unfulfilled yearning, a question posed in rational language that demands a transcendent answer. Field extension is giving voice to these silent yearnings, letting numbers speak themselves into existence.

On Algebraic and Transcendental:

Algebraic elements satisfy polynomial equations—they are numbers that answer to finite algebraic formulas, that can be summoned by finite means. Transcendental elements refuse such constraint. They float free of polynomial grip, answerable to no finite equation, expressible only as limits, infinities, processes without end.

π is transcendental: no polynomial with rational coefficients has π as a root. e is transcendental. Most numbers are transcendental—the algebraic numbers are countable, while the reals are not. Yet we build mathematics on the algebraic, use transcendentals only when infinity forces our hand. Why? Perhaps because the algebraic is that which language can capture—that which answers to names, submits

to finite description. The transcendental is what escapes language, what remains ineffable even as we use it, point to it, calculate with it.

On Finite Fields:

The strangest fields are finite—$\mathbb{Z}_p$ where p is prime, and its extensions. Here is a complete universe with only finitely many elements. Addition and multiplication wrap around, loop back, create patterns of perfect symmetry. These finite fields underlie error-correcting codes, cryptography, the discrete mathematics of computation.

In finite fields, every element has finite multiplicative order—raise it to some power and you return to 1. Every element eventually cycles back to itself. Time becomes circular, transformation becomes rotation, and the infinite is folded into the finite without loss of field structure. These are the most austere mathematical universes: no infinity, no continuity, just discrete elements in perfect algebraic balance.

On Splitting and Completion:

Every polynomial harbors a secret: the field in which it splits completely, where all its roots exist. This splitting field is the polynomial's natural home, the universe it implies. To find it, we adjoin roots one by one, extending the field until the polynomial factors into linear terms, its structure laid bare.

This is mathematics as archaeology: we don't create these fields, we *excavate* them. They were always implicit in the polynomial, always latent in the structure. Field extension is not construction but revelation—removing the veils until the full structure stands exposed.

Greek Wisdom — Ἑλληνική Σοφία

"κάλλος ἀληθείας σπλήνδωρ" — Beauty is the splendor of truth

Plato envisioned ideal forms dwelling beyond the material—perfect, eternal, archetypal. Fields are Platonic forms in precisely this sense: perfect structures that exist in logical space, independent of physical instantiation. Each field is complete, beautiful in its self-sufficiency, true in its internal consistency. The reals don't depend on physical measurement—they would exist as a complete ordered field even in a universe devoid of matter. To study field theory is to contemplate pure form, beauty in its most abstract expression.

Zen Reflection — 禅の省察

The tea bowl holds tea, yet is complete when empty. The field contains elements, yet is perfect in its structure alone.

In Japanese aesthetics, *空間* (kūkan)—empty space—is not absence but presence. The tea bowl's emptiness is its function; its void enables its use. A field is similar: defined not by which elements it contains, but by the operations those elements support, the structure they embody. The rationals and the reals are both fields—different elements, different sizes, yet the same essential structure. Form transcends content. Structure precedes substance. In the empty operations of addition and multiplication, all fields are born.

Enter these self-contained worlds. Each field is a universe sufficient unto itself, yet each opens onto others through extension. In their nested infinities, discover how completion gives rise to incompleteness, how the finite can imply the infinite, how algebraic structure dreams new numbers into existence.

Chapter 11: Field Basics

❦

Fields represent the pinnacle of algebraic structure—spaces where all nonzero elements possess multiplicative inverses, allowing both addition and multiplication to flow with perfect symmetry. From the rational numbers to complex numbers, from finite fields to algebraic closures, fields provide the foundation for understanding polynomial equations, algebraic extensions, and the profound insights of Galois theory.

11.1 Field Definition and Properties

> **Definition**
>
> A **field** is a commutative ring $(F, +, \cdot)$ in which every nonzero element has a multiplicative inverse. Explicitly, F satisfies:
>
> 1. $(F, +)$ is an abelian group with identity 0
> 2. $(F \setminus \{0\}, \cdot)$ is an abelian group with identity 1
> 3. Multiplication distributes over addition: $a(b + c) = ab + ac$ for all $a, b, c \in F$
>
> We denote the multiplicative group of nonzero elements by $F^* = F \setminus \{0\}$.

Fields are precisely the commutative rings with exactly two ideals: (0) and $(1) = F$. This maximally simple ideal structure ensures that fields have no nontrivial quotients.

Theorem

Let F be a field. Then:

1. F has no zero divisors
2. F is an integral domain
3. Every nonzero element of F is a unit
4. The only ideals of F are (0) and F

Proof. (1) If $ab = 0$ with $a \neq 0$, then
$b = 1 \cdot b = (a^{-1}a)b = a^{-1}(ab) = a^{-1} \cdot 0 = 0$.
(2) Since F is commutative with $1 \neq 0$ and has no zero divisors, it is an integral domain.
(3) By definition, every $a \in F^*$ has an inverse $a^{-1} \in F^*$.
(4) If I is a nonzero ideal and $a \in I$ with $a \neq 0$, then $1 = a^{-1} \cdot a \in I$, so $I = F$.
∎

Example

(Rational Numbers) The set $\mathbb{Q} = \{p/q : p, q \in \mathbb{Z}, q \neq 0\}$ forms a field under ordinary addition and multiplication. Every nonzero p/q has inverse q/p.

Example

(Real and Complex Numbers) The real numbers $\mathbb{R}$ and complex numbers $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$ are fields. For $z = a + bi \neq 0$, we have $z^{-1} = \frac{a - bi}{a^2 + b^2}$.

Example

(Finite Fields) For prime p, the set $\mathbb{Z}_p = \{0, 1, 2, \ldots, p-1\}$ with addition and multiplication modulo p forms a field. Since p is prime, every nonzero element a satisfies $\gcd(a, p) = 1$, guaranteeing a multiplicative inverse.

Example

(Constructing $\mathbb{Q}(\sqrt{2})$) Consider $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$. This is a field under the natural operations. For $\alpha = a + b\sqrt{2} \neq 0$, the inverse is:

$$\alpha^{-1} = \frac{a - b\sqrt{2}}{a^2 - 2b^2}$$

provided $a^2 - 2b^2 \neq 0$ (which holds since $\sqrt{2} \notin \mathbb{Q}$).

Example

(The Field $\mathbb{Q}(i)$) The Gaussian rationals $\mathbb{Q}(i) = \{a + bi : a, b \in \mathbb{Q}\}$ form a field with $(a + bi)^{-1} = \frac{a - bi}{a^2 + b^2}$ for $a + bi \neq 0$.

Non-Examples

Not every ring is a field:

Example

(Integers) $\mathbb{Z}$ is not a field since $2 \in \mathbb{Z}$ has no multiplicative inverse in $\mathbb{Z}$ (we would need $1/2$, which is not an integer).

Example

(Composite Moduli) $\mathbb{Z}_6$ is not a field since $2 \cdot 3 \equiv 0 \pmod{6}$, so 2 and 3 are zero divisors without inverses.

Example

(Polynomial Rings) $\mathbb{Q}[x]$ is not a field since x has no multiplicative inverse (we would need $1/x$, which is not a polynomial).

Subfields

Definition

A **subfield** of a field F is a subset $K \subseteq F$ that is itself a field under the inherited operations. We write $K \subseteq F$ or $K \leq F$.

Proposition

A subset K of a field F is a subfield if and only if:

1. $0, 1 \in K$
2. K is closed under addition, multiplication, and taking additive and multiplicative inverses

Equivalently, $K \subseteq F$ is a subfield if it is a subring containing the multiplicative inverses of all its nonzero elements.

Example

We have the tower of subfields: $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$, and $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}) \subset \mathbb{R}$.

Field Homomorphisms

Definition

A **field homomorphism** is a ring homomorphism $\varphi : F \to K$ between fields. That is, φ preserves addition, multiplication, and the multiplicative identity: $\varphi(1_F) = 1_K$.

Theorem

Every field homomorphism is injective.

> ***Proof.*** Let $\varphi : F \to K$ be a field homomorphism. The kernel $\ker(\varphi)$ is an ideal of F. Since F is a field, its only ideals are (0) and F. If $\ker(\varphi) = F$, then $\varphi(1) = 0$, contradicting that φ preserves the identity. Thus $\ker(\varphi) = (0)$, so φ is injective. ∎

This remarkable property distinguishes field homomorphisms from general ring homomorphisms. Between fields, "homomorphism" and "monomorphism" are synonymous.

Field homomorphisms are either the zero map or isomorphisms onto their image.

11.2 Characteristic of a Field

> **Definition**
>
> The **characteristic** of a field F, denoted $\operatorname{char}(F)$, is the smallest positive integer n such that $\underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} = 0$. If no such n exists, we say $\operatorname{char}(F) = 0$.

Equivalently, the characteristic is the order of 1 in the additive group $(F, +)$.

> **Theorem**
>
> The characteristic of a field is either 0 or a prime number.

> ***Proof.*** Suppose $\operatorname{char}(F) = n > 0$ is composite, say $n = ab$ with $1 < a, b < n$. Then in F:
>
> $$0 = \underbrace{1 + 1 + \cdots + 1}_{n} = \underbrace{(1 + \cdots + 1)}_{a} \cdot \underbrace{(1 + \cdots + 1)}_{b}$$
>
> Let $\alpha = \underbrace{1 + \cdots + 1}_{a}$ and $\beta = \underbrace{1 + \cdots + 1}_{b}$. Then $\alpha\beta = 0$ with $\alpha, \beta \neq 0$ (by minimality of n), contradicting that F has no zero divisors. ∎

Example

$\mathrm{char}(\mathbb{Q}) = \mathrm{char}(\mathbb{R}) = \mathrm{char}(\mathbb{C}) = 0$ since $n \cdot 1 \neq 0$ for any positive integer n.

Example

$\mathrm{char}(\mathbb{Z}_p) = p$ for prime p, since $\underbrace{1 + 1 + \cdots + 1}_{p} \equiv 0 \pmod{p}$.

Example

$\mathrm{char}(\mathbb{F}_4) = 2$ where $\mathbb{F}_4$ is the field with 4 elements (constructed as $\mathbb{Z}_2[x]/(x^2 + x + 1)$).

Prime Subfield

Definition

The **prime subfield** of F is the intersection of all subfields of F. It is the smallest subfield of F.

Theorem

The prime subfield of F is isomorphic to:

- $\mathbb{Q}$ if $\mathrm{char}(F) = 0$
- $\mathbb{Z}_p$ if $\mathrm{char}(F) = p > 0$

Proof. Define $\varphi : \mathbb{Z} \to F$ by $\varphi(n) = n \cdot 1_F$. This is a ring homomorphism. If $\mathrm{char}(F) = 0$, then $\ker(\varphi) = \{0\}$ and φ extends to an embedding $\mathbb{Q} \hookrightarrow F$ by $\varphi(m/n) = (m \cdot 1_F)(n \cdot 1_F)^{-1}$. If $\mathrm{char}(F) = p$, then $\ker(\varphi) = p\mathbb{Z}$, inducing an embedding $\mathbb{Z}_p \hookrightarrow F$. ∎

Proposition

In a field of characteristic $p > 0$:

1. $p \cdot a = 0$ for all $a \in F$
2. $(a+b)^p = a^p + b^p$ (Freshman's Dream)
3. The Frobenius map $\varphi(a) = a^p$ is a field homomorphism

Proof. (1) $p \cdot a = (p \cdot 1) \cdot a = 0 \cdot a = 0$.
(2) By the binomial theorem, $(a+b)^p = \sum_{k=0}^{p} \binom{p}{k} a^k b^{p-k}$. For $0 < k < p$, we have $p \mid \binom{p}{k}$, so $\binom{p}{k} \cdot 1 = 0$ in F. Thus $(a+b)^p = a^p + b^p$.
(3) $\varphi(ab) = (ab)^p = a^p b^p = \varphi(a)\varphi(b)$ and
$\varphi(a+b) = a^p + b^p = \varphi(a) + \varphi(b)$ by (2). ∎

11.3 Field of Fractions

Definition

Let R be an integral domain. The **field of fractions** (or **field of quotients**) of R, denoted $\mathrm{Frac}(R)$ or $\mathrm{Quot}(R)$, is a field containing R such that every element of $\mathrm{Frac}(R)$ can be written as a/b with $a, b \in R$ and $b \neq 0$.

Construction

Define an equivalence relation on $R \times (R \setminus \{0\})$ by:

$$(a, b) \sim (c, d) \iff ad = bc$$

This is well-defined since R is an integral domain. Denote the equivalence class of (a, b) by a/b or $\frac{a}{b}$.

Define operations:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}, \qquad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Theorem

With these operations, $\mathrm{Frac}(R)$ is a field, and the map $\iota : R \to \mathrm{Frac}(R)$ given by $\iota(a) = a/1$ is an injective ring homomorphism.

Example

$\mathrm{Frac}(\mathbb{Z}) = \mathbb{Q}$. Every rational number is a quotient of integers.

Example

$\mathrm{Frac}(\mathbb{Q}[x]) = \mathbb{Q}(x)$, the field of rational functions with coefficients in $\mathbb{Q}$. Elements have the form $\frac{p(x)}{q(x)}$ with $p, q \in \mathbb{Q}[x]$ and $q \neq 0$.

Example

$\mathrm{Frac}(\mathbb{Z}[i]) = \mathbb{Q}(i)$. The Gaussian rationals are quotients of Gaussian integers.

Universal Property

Theorem

Let R be an integral domain and F a field. If $\varphi : R \to F$ is an injective ring homomorphism, then there exists a unique field homomorphism $\widetilde{\varphi} : \mathrm{Frac}(R) \to F$ extending φ.

> ***Proof.*** Define $\widetilde{\varphi}(a/b) = \varphi(a) \cdot \varphi(b)^{-1}$. This is well-defined: if $a/b = c/d$, then $ad = bc$, so $\varphi(a)\varphi(d) = \varphi(b)\varphi(c)$, giving $\varphi(a)\varphi(b)^{-1} = \varphi(c)\varphi(d)^{-1}$. Verification that $\widetilde{\varphi}$ is a field homomorphism is routine. Uniqueness follows from the requirement that $\widetilde{\varphi}(a/b) = \widetilde{\varphi}(a) \cdot \widetilde{\varphi}(b)^{-1} = \varphi(a) \cdot \varphi(b)^{-1}$. ∎

The field of fractions is unique up to isomorphism.

11.4 Field Extensions: First Look

> **Definition**
>
> A **field extension** is a pair of fields $F \subseteq K$ where F is a subfield of K. We write K/F and say "K over F" (though this notation does not denote a quotient).

> **Example**
>
> $\mathbb{Q} \subseteq \mathbb{R}$ is a field extension, denoted $\mathbb{R}/\mathbb{Q}$.

> **Example**
>
> $\mathbb{R} \subseteq \mathbb{C}$ gives the extension $\mathbb{C}/\mathbb{R}$.

> **Example**
>
> $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2})$ where $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$.

Simple Extensions

> **Definition**
>
> An extension K/F is **simple** if $K = F(\alpha)$ for some $\alpha \in K$, where $F(\alpha)$ is the smallest subfield of K containing F and α.

The notation $F(\alpha)$ denotes the field obtained by adjoining α to F.

> **Example**
>
> $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is a simple extension generated by $\sqrt{2}$.

Example

$\mathbb{Q}(i)/\mathbb{Q}$ is simple, generated by $i = \sqrt{-1}$.

Example

$\mathbb{C}/\mathbb{R}$ is simple: $\mathbb{C} = \mathbb{R}(i)$.

Tower of Fields

If $F \subseteq E \subseteq K$, we have a **tower** of field extensions: $K/E/F$. Properties of K/F relate to those of K/E and E/F.

Example

Consider the tower $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

11.5 Vector Space Structure

Every field extension K/F can be viewed as a vector space over F, with scalar multiplication given by field multiplication.

Proposition

If K/F is a field extension, then K is a vector space over F with:

- Vector addition: the addition in K
- Scalar multiplication: for $a \in F$ and $\alpha \in K$, define $a \cdot \alpha$ as the product in K

Definition

The **degree** of the extension K/F, denoted $[K : F]$, is the dimension of K as a vector space over F:

$$[K : F] = \dim_F K$$

If $[K : F] < \infty$, we say K/F is a **finite extension**; otherwise, it is an **infinite extension**.

Example

$[\mathbb{C} : \mathbb{R}] = 2$ since $\{1, i\}$ is a basis for $\mathbb{C}$ over $\mathbb{R}$. Every $z \in \mathbb{C}$ can be uniquely written as $z = a + bi$ with $a, b \in \mathbb{R}$.

Example

$[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$ with basis $\{1, \sqrt{2}\}$. Every element of $\mathbb{Q}(\sqrt{2})$ has the form $a + b\sqrt{2}$ with $a, b \in \mathbb{Q}$.

Example

$[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$ with basis $\{1, \sqrt[3]{2}, \sqrt[3]{4}\}$. This follows because the minimal polynomial of $\sqrt[3]{2}$ over $\mathbb{Q}$ is $x^3 - 2$, which has degree 3.

Example

$[\mathbb{Q}(x) : \mathbb{Q}] = \infty$ where $\mathbb{Q}(x)$ is the field of rational functions. The set $\{1, x, x^2, x^3, \ldots\}$ is linearly independent over $\mathbb{Q}$, so the dimension is infinite.

Example

$[\mathbb{F}_4 : \mathbb{F}_2] = 2$ where $\mathbb{F}_4 = \{0, 1, \alpha, \alpha + 1\}$ with $\alpha^2 = \alpha + 1$. A basis is $\{1, \alpha\}$.

Computing Degrees

To compute $[K : F]$, find a basis for K as an F-vector space. Often, adjoining an element α to F creates a basis from powers of α.

> **Example**
>
> To find $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}]$, note that $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2})(\sqrt{3})$. First, $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$. Then $\sqrt{3} \notin \mathbb{Q}(\sqrt{2})$ (provable by supposing $\sqrt{3} = a + b\sqrt{2}$ and deriving a contradiction), so $x^2 - 3$ is irreducible over $\mathbb{Q}(\sqrt{2})$, giving $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{2})] = 2$. By the tower law (next chapter), $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = 2 \cdot 2 = 4$.
> A basis is $\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$.

> **Greek Meditation: Fields as Perfect Realms**
>
> The ancient Greeks sought the realm of perfect Forms—eternal, unchanging, and complete. A field embodies this Platonic ideal in algebra: a space where every question has an answer, every nonzero element possesses an inverse, and the operations of arithmetic achieve perfect symmetry. Just as the circle represents geometric perfection, the field represents algebraic completeness. In this perfect realm, division is always possible (except by zero), equations can be solved, and the structure admits no internal contradictions. The field is the Platonic heaven of numbers, where mathematical truth resides in its purest form.

> **Zen Meditation: The Field of Emptiness**
>
> In Zen, emptiness (śūnyatā) is not mere nothingness but the potential for all phenomena. A field similarly embodies perfect emptiness—it contains nothing that prevents any operation. All nonzero elements have inverses; all operations are possible; nothing is lacking. Like the empty mirror reflecting all images, the field structure allows all arithmetic to unfold naturally. There are no obstructions, no zero divisors, no missing inverses. In this mathematical emptiness, everything becomes possible. The field is complete not through accumulation but through the absence of limitation.

Visualizations

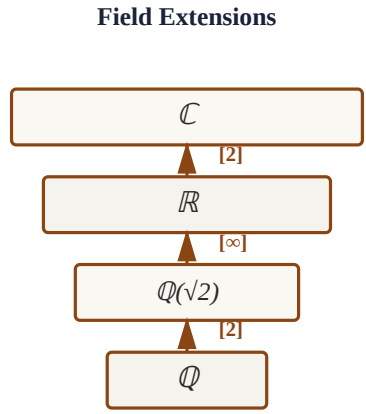


Figure 11.1: Tower of field extensions from $\mathbb{Q}$ to $\mathbb{C}$

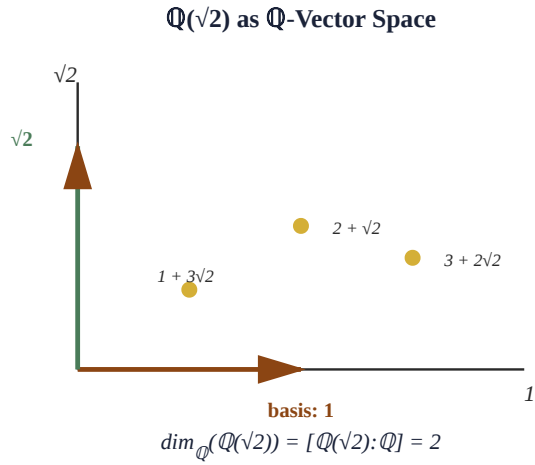


Figure 11.2: $\mathbb{Q}(\sqrt{2})$ as a 2-dimensional vector space over $\mathbb{Q}$ with basis $\{1, \sqrt{2}\}$

Practice Problems

1. Prove that every finite integral domain is a field.

> ***Solution.*** Let R be a finite integral domain and let $a \in R$ with $a \neq 0$. Consider the map $\varphi_a : R \to R$ defined by $\varphi_a(x) = ax$. Since R has no zero divisors, φ_a is injective: if $ax = ay$, then $a(x-y) = 0$, so $x = y$. Since R is finite, an injective map $R \to R$ is also surjective. Thus $1 \in \mathrm{Im}(\varphi_a)$, so there exists $b \in R$ with $ab = 1$. Hence every nonzero element has an inverse, making R a field.

2. Show that $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\}$ is not a field.

> ***Solution.*** Note that $2 \in \mathbb{Z}[\sqrt{-5}]$. If $\mathbb{Z}[\sqrt{-5}]$ were a field, then $1/2$ would belong to it, so $1/2 = a + b\sqrt{-5}$ for some $a, b \in \mathbb{Z}$. Then $1 = 2a + 2b\sqrt{-5}$, which implies $2a = 1$ and $2b = 0$. But $2a = 1$ has no integer solution. Therefore $\mathbb{Z}[\sqrt{-5}]$ is not a field.

3. Prove that if F is a field of characteristic $p > 0$, then $|F| = p^n$ for some $n \geq 1$ if F is finite.

> ***Solution.*** The prime subfield of F is isomorphic to $\mathbb{Z}_p$, so F contains a copy of $\mathbb{Z}_p$. View F as a vector space over $\mathbb{Z}_p$. Since F is finite, it has finite dimension, say $[F : \mathbb{Z}_p] = n$. Then $|F| = |\mathbb{Z}_p|^n = p^n$.

4. Find the inverse of $3 + 2\sqrt{2}$ in $\mathbb{Q}(\sqrt{2})$.

> ***Solution.*** We use rationalization:
> $(3 + 2\sqrt{2})^{-1} = \frac{1}{3+2\sqrt{2}} = \frac{3-2\sqrt{2}}{(3+2\sqrt{2})(3-2\sqrt{2})} = \frac{3-2\sqrt{2}}{9-8} = 3 - 2\sqrt{2}$.
> Verification: $(3 + 2\sqrt{2})(3 - 2\sqrt{2}) = 9 - 8 = 1$. ✓

5. Show that $\mathbb{Q}(\sqrt{2}) \neq \mathbb{Q}(\sqrt{3})$.

> ***Solution.*** Suppose $\mathbb{Q}(\sqrt{2}) = \mathbb{Q}(\sqrt{3})$. Then $\sqrt{3} \in \mathbb{Q}(\sqrt{2})$, so $\sqrt{3} = a + b\sqrt{2}$ for some $a, b \in \mathbb{Q}$. Squaring: $3 = a^2 + 2b^2 + 2ab\sqrt{2}$. Since 1 and $\sqrt{2}$ are linearly independent over $\mathbb{Q}$, we have $a^2 + 2b^2 = 3$ and $2ab = 0$. From $2ab = 0$, either $a = 0$ or $b = 0$. If $b = 0$, then $a^2 = 3$, so $a = \pm\sqrt{3} \notin \mathbb{Q}$. If

> $a = 0$, then $2b^2 = 3$, so $b = \pm\sqrt{3/2} \notin \mathbb{Q}$. Both cases contradict $a, b \in \mathbb{Q}$, so $\mathbb{Q}(\sqrt{2}) \neq \mathbb{Q}(\sqrt{3})$.

6. Determine $[\mathbb{Q}(i) : \mathbb{Q}]$ and find a basis.

> ***Solution.*** Every element of $\mathbb{Q}(i)$ has the form $a + bi$ with $a, b \in \mathbb{Q}$. The set $\{1, i\}$ is linearly independent over $\mathbb{Q}$: if $a + bi = 0$ with $a, b \in \mathbb{Q}$, then comparing real and imaginary parts gives $a = b = 0$. Since $\{1, i\}$ spans $\mathbb{Q}(i)$ and is linearly independent, it forms a basis. Therefore $[\mathbb{Q}(i) : \mathbb{Q}] = 2$.

7. Show that $\mathrm{Frac}(\mathbb{Z}[x]) \cong \mathbb{Q}(x)$.

> ***Solution.*** By definition, $\mathrm{Frac}(\mathbb{Z}[x])$ consists of elements $f(x)/g(x)$ with $f, g \in \mathbb{Z}[x]$ and $g \neq 0$. Similarly, $\mathbb{Q}(x)$ consists of $p(x)/q(x)$ with $p, q \in \mathbb{Q}[x]$ and $q \neq 0$. Every element of $\mathbb{Q}(x)$ can be written as $(a_0 + a_1 x + \cdots)/(b_0 + b_1 x + \cdots)$ with $a_i, b_i \in \mathbb{Q}$. Clearing denominators of the coefficients gives an element of $\mathrm{Frac}(\mathbb{Z}[x])$. Conversely, every element of $\mathrm{Frac}(\mathbb{Z}[x])$ is already in $\mathbb{Q}(x)$. Thus the inclusion $\mathbb{Z}[x] \hookrightarrow \mathbb{Q}[x]$ extends to an isomorphism $\mathrm{Frac}(\mathbb{Z}[x]) \cong \mathbb{Q}(x)$.

8. Prove that $\mathbb{Q}$ has no proper finite extensions.

> ***Solution.*** False statement. $\mathbb{Q}$ has many proper finite extensions, such as $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt[3]{2})$, etc. The problem likely meant to ask something else. If the question is "Prove $\mathbb{Q}$ has infinitely many finite extensions," then: For each prime p, the extension $\mathbb{Q}(\sqrt{p})$ is a degree 2 extension of $\mathbb{Q}$, and $\mathbb{Q}(\sqrt{p}) \neq \mathbb{Q}(\sqrt{q})$ for distinct primes p, q (provable by a similar argument to Problem 5). Since there are infinitely many primes, $\mathbb{Q}$ has infinitely many distinct quadratic extensions.

9. Show that the characteristic of $\mathbb{Q}(\sqrt{2})$ is 0.

> ***Solution.*** Since $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2})$, the characteristic of $\mathbb{Q}(\sqrt{2})$ divides the characteristic of $\mathbb{Q}$. But $\mathrm{char}(\mathbb{Q}) = 0$ (since $n \cdot 1 \neq 0$ for all positive integers n in $\mathbb{Q}$). Therefore $\mathrm{char}(\mathbb{Q}(\sqrt{2})) = 0$.

10. Prove that if F is a finite field, then $\mathrm{char}(F) > 0$.

> ***Solution.*** Suppose $\text{char}(F) = 0$. Then the prime subfield of F is isomorphic to $\mathbb{Q}$, which is infinite. Since $\mathbb{Q} \subseteq F$, we have $|F| \geq |\mathbb{Q}| = \infty$, contradicting that F is finite. Therefore $\text{char}(F) > 0$.

11. Compute $(1+\alpha)^4$ in $\mathbb{F}_4 = \mathbb{F}_2[x]/(x^2+x+1)$ where α is the image of x.

> ***Solution.*** In characteristic 2, $(a+b)^2 = a^2 + b^2$ (Freshman's Dream). Thus $(1+\alpha)^2 = 1^2 + \alpha^2 = 1 + \alpha^2$. We have $\alpha^2 = \alpha + 1$ (from $x^2 + x + 1 \equiv 0$), so $(1+\alpha)^2 = 1 + \alpha + 1 = \alpha$ (since $1 + 1 = 0$ in characteristic 2). Therefore $(1+\alpha)^4 = ((1+\alpha)^2)^2 = \alpha^2 = \alpha + 1$.

12. Show that there is no field with exactly 6 elements.

> ***Solution.*** If F is a finite field, then $|F| = p^n$ for some prime p and positive integer n (as shown in Problem 3). Since $6 = 2 \cdot 3$ is not a prime power, there is no field with exactly 6 elements.

13. Prove that any two finite fields with the same number of elements are isomorphic.

> ***Solution.*** This is a deep result. Both fields have p^n elements for some prime p and positive integer n. Both are splitting fields of $x^{p^n} - x$ over $\mathbb{Z}_p$ (to be shown in later chapters). Since splitting fields are unique up to isomorphism, the two fields are isomorphic. Full proof requires Galois theory.

14. Let F be a field. Show that the polynomial ring $F[x]$ is never a field.

> ***Solution.*** Suppose $F[x]$ is a field. Then $x \in F[x]$ has a multiplicative inverse $p(x) \in F[x]$ with $x \cdot p(x) = 1$. Taking degrees:
> $\deg(x) + \deg(p(x)) = \deg(1)$, so $1 + \deg(p(x)) = 0$, giving
> $\deg(p(x)) = -1$. But the degree of a nonzero polynomial is a non-negative integer, contradiction. Therefore $F[x]$ is not a field.

15. Show that $\mathbb{Z}_2[x]/(x^2+x+1)$ is a field with 4 elements.

> ***Solution.*** Since $x^2 + x + 1$ is irreducible over $\mathbb{Z}_2$ (it has degree 2 and no roots in $\mathbb{Z}_2$: $0^2 + 0 + 1 = 1 \neq 0$ and $1^2 + 1 + 1 = 1 \neq 0$), the quotient $F = \mathbb{Z}_2[x]/(x^2+x+1)$ is a field. Let $\alpha = \overline{x}$ be the image of x in F. Then $F = \{a + b\alpha : a, b \in \mathbb{Z}_2\}$, giving $|F| = 2^2 = 4$ elements: $\{0, 1, \alpha, 1+\alpha\}$.

16. Determine $[\mathbb{Q}(\sqrt{2}+\sqrt{3}):\mathbb{Q}]$.

> ***Solution.*** Let $\alpha = \sqrt{2}+\sqrt{3}$. Then $\alpha^2 = 5+2\sqrt{6}$, so $\sqrt{6} = \frac{\alpha^2-5}{2} \in \mathbb{Q}(\alpha)$. Also, $\alpha = \sqrt{2}+\sqrt{3}$, so $\sqrt{2} = \alpha - \sqrt{3}$. Then $2 = \alpha^2 - 2\alpha\sqrt{3} + 3$, giving $\sqrt{3} = \frac{\alpha^2+1}{2\alpha} \in \mathbb{Q}(\alpha)$. Similarly, $\sqrt{2} = \alpha - \sqrt{3} \in \mathbb{Q}(\alpha)$. Thus $\mathbb{Q}(\sqrt{2},\sqrt{3}) \subseteq \mathbb{Q}(\alpha)$. Since the reverse inclusion is obvious, $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{2},\sqrt{3})$. We know $[\mathbb{Q}(\sqrt{2},\sqrt{3}):\mathbb{Q}] = 4$ (from Example in §11.5), so $[\mathbb{Q}(\sqrt{2}+\sqrt{3}):\mathbb{Q}] = 4$.

17. Prove that if $[K:F]$ is prime, then there are no intermediate fields between F and K.

> ***Solution.*** Suppose $F \subseteq E \subseteq K$. By the tower law (next chapter), $[K:F] = [K:E]\cdot[E:F]$. Since $[K:F]$ is prime and both $[K:E]$ and $[E:F]$ are positive integers, either $[E:F]=1$ or $[K:E]=1$. If $[E:F]=1$, then $E=F$. If $[K:E]=1$, then $K=E$. Thus there are no proper intermediate fields.

18. Show that $\mathbb{C}$ cannot be given the structure of a vector space over $\mathbb{Q}(\sqrt{2})$ such that the scalar multiplication extends the usual multiplication in $\mathbb{C}$.

> ***Solution.*** False premise—actually, $\mathbb{C}$ can be given such a structure since $\mathbb{Q}(\sqrt{2}) \subset \mathbb{R} \subset \mathbb{C}$, and $\mathbb{C}$ is naturally an $\mathbb{R}$-vector space, hence also a $\mathbb{Q}(\sqrt{2})$-vector space by restriction of scalars. The dimension is $[\mathbb{C}:\mathbb{Q}(\sqrt{2})] = \frac{[\mathbb{C}:\mathbb{Q}]}{[\mathbb{Q}(\sqrt{2}):\mathbb{Q}]} = \frac{\infty}{2} = \infty$ (assuming the tower law for infinite extensions, though technically $[\mathbb{C}:\mathbb{Q}]$ is not well-defined as a finite number—it's an infinite-dimensional vector space).

19. Prove that in a field of characteristic 2, $(a+b)^{2^n} = a^{2^n} + b^{2^n}$ for all $n \geq 1$.

> ***Solution.*** Proceed by induction on n. Base case $n=1$: $(a+b)^2 = a^2+b^2$ by Freshman's Dream in characteristic 2. Inductive step: Assume $(a+b)^{2^k} = a^{2^k}+b^{2^k}$. Then:
>
> $$(a+b)^{2^{k+1}} = ((a+b)^{2^k})^2 = (a^{2^k}+b^{2^k})^2 = (a^{2^k})^2 + (b^{2^k})^2 = a^{2^{k+1}} + b^{2^{k+1}}$$
>
> where the third equality uses the base case applied to a^{2^k} and b^{2^k}.

20. Show that every field has infinitely many field extensions.

Solution. Let F be any field. For each positive integer n, consider the polynomial ring $F[x]$ and an irreducible polynomial $f(x)$ of degree n (such polynomials exist—for example, in characteristic 0, we can use cyclotomic polynomials, or more generally, use the fact that there exist irreducible polynomials of every degree). The quotient $K_n = F[x]/(f(x))$ is a field extension of F with $[K_n : F] = n$. Since extensions of different degrees are non-isomorphic (over F), we obtain infinitely many distinct field extensions of F.

✦

End of Chapter 11

Chapter 12: Field Extensions

The study of field extensions reveals how larger fields emerge from smaller ones, capturing the essence of adjoining roots and building solution spaces for polynomials. Through the tower law, minimal polynomials, and the distinction between algebraic and transcendental elements, we gain powerful tools for understanding the intricate structure of field hierarchies.

12.1 Degree of Extensions

> **Definition**
>
> The **degree** of a field extension K/F, denoted $[K : F]$, is the dimension of K as a vector space over F. We say K/F is a **finite extension** if $[K : F] < \infty$, and an **infinite extension** otherwise.

> **Example**
>
> $[\mathbb{C} : \mathbb{R}] = 2$ with basis $\{1, i\}$ over $\mathbb{R}$.

> **Example**
>
> $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$ with basis $\{1, \sqrt{2}\}$.

Example

$[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$ with basis $\{1, \sqrt[3]{2}, (\sqrt[3]{2})^2\}$.

Example

$[\mathbb{R} : \mathbb{Q}] = \infty$. The set $\{1, \sqrt{2}, \sqrt{3}, \sqrt{5}, \ldots\}$ is linearly independent over $\mathbb{Q}$.

Example

$[\mathbb{Q}(x) : \mathbb{Q}] = \infty$ where $\mathbb{Q}(x)$ is the field of rational functions. The set $\{1, x, x^2, x^3, \ldots\}$ is linearly independent.

Proposition

If K/F is a finite extension with $[K : F] = n$, then:

1. Every element $\alpha \in K$ satisfies a nonzero polynomial equation of degree at most n with coefficients in F
2. Any $n + 1$ elements of K are linearly dependent over F

Proof. (1) Consider the $n + 1$ elements $1, \alpha, \alpha^2, \ldots, \alpha^n \in K$. Since $\dim_F K = n$, these are linearly dependent, so there exist $a_0, \ldots, a_n \in F$ (not all zero) with $\sum_{i=0}^{n} a_i \alpha^i = 0$.
(2) If $\alpha_1, \ldots, \alpha_{n+1} \in K$, they cannot be linearly independent since $\dim_F K = n$. ∎

12.2 Tower Law

Theorem

(Tower Law / Multiplicativity of Degrees) If $F \subseteq E \subseteq K$ is a tower of field extensions, then:

$$[K : F] = [K : E] \cdot [E : F]$$

In particular, if $[K : E]$ and $[E : F]$ are both finite, then $[K : F]$ is finite.

Proof. Suppose $[E : F] = m$ and $[K : E] = n$ with bases $\{\alpha_1, \ldots, \alpha_m\}$ for E over F and $\{\beta_1, \ldots, \beta_n\}$ for K over E. We claim that $\{\alpha_i \beta_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ is a basis for K over F.

Spanning: Let $\gamma \in K$. Since $\{\beta_j\}$ spans K over E, we have $\gamma = \sum_{j=1}^{n} e_j \beta_j$ with $e_j \in E$. Since $\{\alpha_i\}$ spans E over F, each $e_j = \sum_{i=1}^{m} f_{ij} \alpha_i$ with $f_{ij} \in F$. Thus $\gamma = \sum_{i,j} f_{ij} \alpha_i \beta_j$.

Linear Independence: Suppose $\sum_{i,j} f_{ij} \alpha_i \beta_j = 0$ with $f_{ij} \in F$. Rewrite as $\sum_j (\sum_i f_{ij} \alpha_i) \beta_j = 0$. Since $\{\beta_j\}$ is linearly independent over E, we have $\sum_i f_{ij} \alpha_i = 0$ for all j. Since $\{\alpha_i\}$ is linearly independent over F, we have $f_{ij} = 0$ for all i, j.

Therefore $[K : F] = mn = [K : E] \cdot [E : F]$. ∎

If K/F is a finite extension and $\alpha \in K$, then $[F(\alpha) : F]$ divides $[K : F]$.

Proof. Apply the tower law to $F \subseteq F(\alpha) \subseteq K$: $[K : F] = [K : F(\alpha)] \cdot [F(\alpha) : F]$. ∎

Example

Consider $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt{2}, \sqrt{3})$. We have $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$ and $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{2})] = 2$ (since $x^2 - 3$ is irreducible over $\mathbb{Q}(\sqrt{2})$).
By the tower law: $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = 2 \cdot 2 = 4$.
A basis is $\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$.

Example

Consider $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt[3]{2}) \subseteq \mathbb{Q}(\sqrt[3]{2}, \omega)$ where $\omega = e^{2\pi i/3}$ is a primitive cube root of unity. We have $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$ and $[\mathbb{Q}(\sqrt[3]{2}, \omega) : \mathbb{Q}(\sqrt[3]{2})] = 2$ (since ω satisfies $x^2 + x + 1 = 0$, which is irreducible over $\mathbb{Q}(\sqrt[3]{2})$ as $\omega \notin \mathbb{R}$). Thus $[\mathbb{Q}(\sqrt[3]{2}, \omega) : \mathbb{Q}] = 6$.

If $[K : F]$ is prime, then there are no intermediate fields between F and K (i.e., no E with $F \subsetneq E \subsetneq K$).

> ***Proof.*** If $F \subseteq E \subseteq K$, then $[K : F] = [K : E] \cdot [E : F]$. Since $[K : F]$ is prime, either $[E : F] = 1$ (so $E = F$) or $[K : E] = 1$ (so $K = E$). ∎

12.3 Simple Extensions

Definition

An extension K/F is **simple** if there exists $\alpha \in K$ such that $K = F(\alpha)$, where $F(\alpha)$ denotes the smallest subfield of K containing F and α.

Explicitly, $F(\alpha)$ consists of all elements that can be expressed using α and elements of F via field operations (addition, subtraction, multiplication, division).

Proposition

For $\alpha \in K$ with K/F a field extension:

$$F(\alpha) = \left\{ \frac{f(\alpha)}{g(\alpha)} : f, g \in F[x], g(\alpha) \neq 0 \right\}$$

That is, $F(\alpha)$ consists of all rational functions in α with coefficients in F.

Definition

The ring $F[\alpha]$ is the smallest subring of K containing F and α:

$F[\alpha] = \{f(\alpha) : f \in F[x]\} = \{a_0 + a_1\alpha + \cdots + a_n\alpha^n : n \geq 0, a_i \in F\}$

Note that $F[\alpha] \subseteq F(\alpha)$ always, but equality does not always hold.

Proposition

The following are equivalent:

1. $F[\alpha] = F(\alpha)$
2. $F[\alpha]$ is a field
3. α is algebraic over F (i.e., α is a root of some nonzero polynomial in $F[x]$)

Example

For $\alpha = \sqrt{2}$: $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\} = \mathbb{Q}(\sqrt{2})$ since $\sqrt{2}$ is algebraic.

Example

For $\alpha = x$ (an indeterminate): $\mathbb{Q}[x] = \{$ polynomials in $x\} \neq \mathbb{Q}(x) = \{$ rational functions in $x\}$. Here x is transcendental over $\mathbb{Q}$.

12.4 Algebraic vs Transcendental Elements

Definition

Let K/F be a field extension and $\alpha \in K$.

- α is **algebraic over** F if there exists a nonzero polynomial $f \in F[x]$ such that $f(\alpha) = 0$.
- α is **transcendental over** F if α is not algebraic over F.

Example

$\sqrt{2}$ is algebraic over $\mathbb{Q}$ since $x^2 - 2 = 0$ at $x = \sqrt{2}$.

Example

$i = \sqrt{-1}$ is algebraic over $\mathbb{R}$ since $x^2 + 1 = 0$ at $x = i$.

Example

π and e are transcendental over $\mathbb{Q}$ (deep results proved by Lindemann and Hermite, respectively).

Example

Every element of $\mathbb{C}$ is algebraic over $\mathbb{R}$, since if $z = a + bi \in \mathbb{C}$, then z satisfies $(x - z)(x - \bar{z}) = x^2 - 2ax + (a^2 + b^2) \in \mathbb{R}[x]$.

Theorem

Let $\alpha \in K$ where K/F is a field extension. The following are equivalent:

1. α is algebraic over F
2. $[F(\alpha) : F] < \infty$
3. $F[\alpha] = F(\alpha)$
4. $F[\alpha]$ is a field

> ***Proof.*** $(1 \Rightarrow 2)$: If α is algebraic, let $f \in F[x]$ be a nonzero polynomial of minimal degree n with $f(\alpha) = 0$. Then $\{1, \alpha, \alpha^2, \ldots, \alpha^{n-1}\}$ spans $F(\alpha)$ over F (since α^n can be expressed in terms of lower powers using $f(\alpha) = 0$). Thus $[F(\alpha) : F] \leq n < \infty$.
>
> $(2 \Rightarrow 1)$: If $[F(\alpha) : F] = n < \infty$, then $\{1, \alpha, \alpha^2, \ldots, \alpha^n\}$ are $n+1$ elements in an n-dimensional space, hence linearly dependent. Thus there exist $a_0, \ldots, a_n \in F$ (not all zero) with $\sum a_i \alpha^i = 0$, so α is algebraic.
>
> $(1 \Rightarrow 3)$: If α is algebraic with minimal polynomial f of degree n, every element of $F[\alpha]$ can be written as a polynomial in α of degree $< n$. For $\beta = g(\alpha) \in F[\alpha]$ with $\beta \neq 0$, we have $\gcd(f, g) = 1$ (since f is irreducible and $g(\alpha) \neq 0$), so by Bézout there exist $u, v \in F[x]$ with $uf + vg = 1$. Evaluating at α: $v(\alpha)g(\alpha) = 1$, so $\beta^{-1} = v(\alpha) \in F[\alpha]$. Thus $F[\alpha]$ is a field, so $F[\alpha] = F(\alpha)$.
>
> $(3 \Rightarrow 4)$: Clear.
>
> $(4 \Rightarrow 1)$: If $F[\alpha]$ is a field and $\alpha \neq 0$, then $\alpha^{-1} \in F[\alpha]$, so $\alpha^{-1} = a_0 + a_1\alpha + \cdots + a_n\alpha^n$ for some $a_i \in F$. Thus $1 = \alpha(a_0 + a_1\alpha + \cdots + a_n\alpha^n)$, giving $a_0\alpha + a_1\alpha^2 + \cdots + a_n\alpha^{n+1} - 1 = 0$. Hence α satisfies a polynomial equation with coefficients in F. $\blacksquare$

If α is transcendental over F, then $F(\alpha) \cong F(x)$ (the field of rational functions) and $[F(\alpha) : F] = \infty$.

> ***Proof.*** The evaluation map $\varphi : F[x] \to F[\alpha]$ given by $\varphi(f) = f(\alpha)$ is a surjective homomorphism. If α is transcendental, $\ker(\varphi) = \{0\}$, so φ is an isomorphism $F[x] \cong F[\alpha]$. This extends to an isomorphism $F(x) \cong F(\alpha)$ of fraction fields. $\blacksquare$

12.5 Minimal Polynomials

Definition

Let $\alpha \in K$ be algebraic over F. The **minimal polynomial** of α over F, denoted $m_{\alpha,F}(x)$ or simply $m_\alpha(x)$, is the unique monic polynomial of minimal degree in $F[x]$ that has α as a root.

Theorem

Let $\alpha \in K$ be algebraic over F with minimal polynomial $m(x)$. Then:

1. $m(x)$ is irreducible in $F[x]$
2. If $f \in F[x]$ satisfies $f(\alpha) = 0$, then $m(x)$ divides $f(x)$
3. $m(x)$ is the unique monic irreducible polynomial in $F[x]$ having α as a root
4. $\deg(m) = [F(\alpha) : F]$
5. $F(\alpha) \cong F[x]/(m(x))$

Proof. (1) Suppose $m(x) = g(x)h(x)$ with $\deg(g), \deg(h) < \deg(m)$. Then $0 = m(\alpha) = g(\alpha)h(\alpha)$. Since $F(\alpha)$ is a field, $g(\alpha) = 0$ or $h(\alpha) = 0$. But this contradicts minimality of $\deg(m)$. Thus m is irreducible.

(2) Divide: $f(x) = q(x)m(x) + r(x)$ with $\deg(r) < \deg(m)$. Then $0 = f(\alpha) = q(\alpha)m(\alpha) + r(\alpha) = r(\alpha)$. By minimality, $r = 0$, so $m \mid f$.

(3) Uniqueness follows from (2) and monic normalization.

(4) If $\deg(m) = n$, then $\{1, \alpha, \ldots, \alpha^{n-1}\}$ is a basis for $F(\alpha)$ over F (linear independence from minimality, spanning from being able to reduce α^k for $k \geq n$ using $m(\alpha) = 0$).

(5) The evaluation map $F[x] \to F[\alpha] = F(\alpha)$ has kernel $(m(x))$ by (2), inducing an isomorphism $F[x]/(m(x)) \cong F(\alpha)$. ∎

Example

The minimal polynomial of $\sqrt{2}$ over $\mathbb{Q}$ is $m(x) = x^2 - 2$. This is monic, irreducible (by Eisenstein or checking it has no rational roots), and $m(\sqrt{2}) = 0$. We have $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = \deg(m) = 2$.

Example

The minimal polynomial of $\sqrt[3]{2}$ over $\mathbb{Q}$ is $m(x) = x^3 - 2$ (irreducible by Eisenstein with $p = 2$). Thus $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$.

Example

The minimal polynomial of i over $\mathbb{R}$ is $m(x) = x^2 + 1$. Thus $[\mathbb{R}(i) : \mathbb{R}] = [\mathbb{C} : \mathbb{R}] = 2$.

Example

The minimal polynomial of i over $\mathbb{Q}$ is also $m(x) = x^2 + 1$ (irreducible over $\mathbb{Q}$ since it has no rational roots). Thus $[\mathbb{Q}(i) : \mathbb{Q}] = 2$.

Example

Find the minimal polynomial of $\alpha = \sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$.
We have $\alpha = \sqrt{2} + \sqrt{3}$, so $\alpha - \sqrt{2} = \sqrt{3}$. Squaring: $\alpha^2 - 2\alpha\sqrt{2} + 2 = 3$, giving $\alpha^2 - 1 = 2\alpha\sqrt{2}$. Squaring again: $(\alpha^2 - 1)^2 = 4\alpha^2 \cdot 2 = 8\alpha^2$, so $\alpha^4 - 2\alpha^2 + 1 = 8\alpha^2$, thus $\alpha^4 - 10\alpha^2 + 1 = 0$.
The polynomial $m(x) = x^4 - 10x^2 + 1$ is irreducible over $\mathbb{Q}$ (can be verified using various tests), so this is the minimal polynomial. Thus $[\mathbb{Q}(\sqrt{2} + \sqrt{3}) : \mathbb{Q}] = 4$.

Example

Find the minimal polynomial of $\omega = e^{2\pi i/3}$ over $\mathbb{Q}$.
We have $\omega^3 = 1$ and $\omega \neq 1$, so $\omega^3 - 1 = 0$. Factoring: $(x - 1)(x^2 + x + 1) = 0$. Since $\omega \neq 1$, we have $\omega^2 + \omega + 1 = 0$. The polynomial $x^2 + x + 1$ is irreducible over $\mathbb{Q}$ (no rational roots), so $m_\omega(x) = x^2 + x + 1$.

Greek Meditation: Transcendence and the Ineffable

> The ancient Greeks encountered the ineffable—quantities that transcended rational understanding. The ratio of a circle's circumference to its diameter, the base of natural logarithms: these transcendental numbers cannot be captured by any polynomial equation with rational coefficients. They stand outside the algebraic realm, beyond the reach of finite algebraic operations. Like Plato's Form of the Good, which transcends even Being itself, transcendental numbers transcend the algebraic hierarchy. They remind us that reality extends beyond what can be constructed through finite means, that some truths lie forever beyond the grasp of algebraic reason. In mathematics as in philosophy, transcendence signals the limits of systematic knowledge and the presence of the truly infinite.

> **Zen Meditation: The Minimal Polynomial**
>
> The minimal polynomial embodies the Zen principle of simplicity as essence. Just as a Zen koan uses the fewest words to point toward enlightenment, the minimal polynomial is the most economical algebraic expression that captures the nature of an element. It contains no extraneous information, no unnecessary complexity—only what is essential. To find the minimal polynomial is to strip away all that is superfluous, revealing the element in its purest algebraic form. Like the haiku's seventeen syllables or the ink painting's sparse brush strokes, the minimal polynomial achieves completeness through radical simplicity. It is the algebraic manifestation of Zen's teaching: less is more, simplicity is depth, essence requires no elaboration.

❦

Visualizations

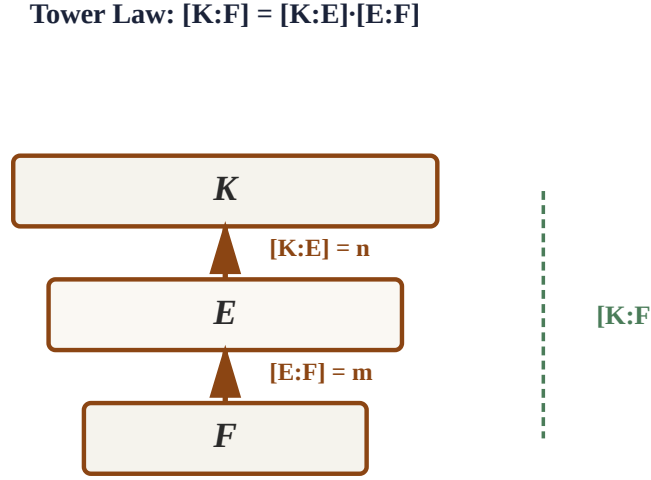


Figure 12.1: Tower Law illustration

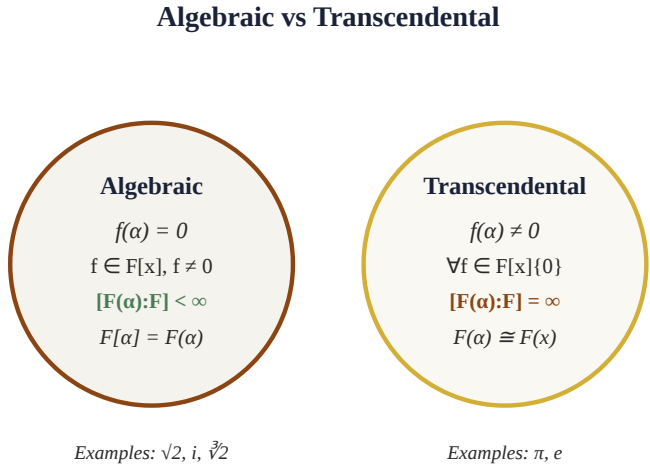


Figure 12.2: Comparison of algebraic and transcendental elements

Practice Problems

1. Prove that if $[K : F]$ is prime, then $K = F(\alpha)$ for any $\alpha \in K \setminus F$.

> ***Solution.*** Let $\alpha \in K \setminus F$. Then $F \subsetneq F(\alpha) \subseteq K$, so by the tower law $[K : F] = [K : F(\alpha)] \cdot [F(\alpha) : F]$. Since $[K : F]$ is prime and $[F(\alpha) : F] > 1$ (as $\alpha \notin F$), we must have $[F(\alpha) : F] = [K : F]$ and $[K : F(\alpha)] = 1$. Thus $K = F(\alpha)$.

2. Show that $[\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}) : \mathbb{Q}] = 8$.

> ***Solution.*** Consider the tower
> $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt{2}, \sqrt{3}) \subseteq \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$.
> • $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$
> • $x^2 - 3$ is irreducible over $\mathbb{Q}(\sqrt{2})$ (if $\sqrt{3} = a + b\sqrt{2}$, squaring gives contradiction), so $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{2})] = 2$
> • $x^2 - 5$ is irreducible over $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ (similar argument), so $[\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}) : \mathbb{Q}(\sqrt{2}, \sqrt{3})] = 2$
> By tower law: $[\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}) : \mathbb{Q}] = 2 \cdot 2 \cdot 2 = 8$.

3. Find the minimal polynomial of $\omega = e^{2\pi i/4} = \frac{1+i}{\sqrt{2}}$ over $\mathbb{Q}$.

> ***Solution.*** We have $\omega^4 = 1$ and $\omega \neq \pm 1$, so ω is a root of $x^4 - 1 = (x - 1)(x + 1)(x^2 + 1)$. Since $\omega \neq \pm 1$, we have $\omega^2 + 1 = 0$. The polynomial $x^2 + 1$ is irreducible over $\mathbb{Q}$ (no rational roots), so $m_\omega(x) = x^2 + 1$. Thus $[\mathbb{Q}(\omega) : \mathbb{Q}] = 2$. (Note: $\omega = \frac{1+i}{\sqrt{2}}$, so $\mathbb{Q}(\omega) = \mathbb{Q}(i)$.)

4. Prove that if α is algebraic over F and β is algebraic over $F(\alpha)$, then β is algebraic over F.

> ***Solution.*** Since β is algebraic over $F(\alpha)$, we have $[F(\alpha)(\beta) : F(\alpha)] < \infty$. Since α is algebraic over F, we have $[F(\alpha) : F] < \infty$. By the tower law:

$[F(\alpha,\beta):F] = [F(\alpha,\beta):F(\alpha)]\cdot[F(\alpha):F] < \infty$. Since $F \subseteq F(\beta) \subseteq F(\alpha,\beta)$ and $[F(\alpha,\beta):F] < \infty$, we have $[F(\beta):F] < \infty$ (divides $[F(\alpha,\beta):F]$). Therefore β is algebraic over F.

5. Show that $\mathbb{Q}(\sqrt{2+\sqrt{2}}) = \mathbb{Q}(\sqrt{2})$.

Solution. Let $\alpha = \sqrt{2+\sqrt{2}}$. Then $\alpha^2 = 2+\sqrt{2}$, so $\sqrt{2} = \alpha^2 - 2 \in \mathbb{Q}(\alpha)$. Thus $\mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\alpha)$.
Conversely, $\alpha = \sqrt{2+\sqrt{2}} \in \mathbb{Q}(\sqrt{2})$ since we can express it using elements of $\mathbb{Q}(\sqrt{2})$. Thus $\mathbb{Q}(\alpha) \subseteq \mathbb{Q}(\sqrt{2})$.
Therefore $\mathbb{Q}(\sqrt{2+\sqrt{2}}) = \mathbb{Q}(\sqrt{2})$.

6. Determine $[\mathbb{Q}(\sqrt[4]{2}):\mathbb{Q}]$ and find the minimal polynomial of $\sqrt[4]{2}$ over $\mathbb{Q}$.

Solution. The minimal polynomial is $m(x) = x^4 - 2$ (irreducible over $\mathbb{Q}$ by Eisenstein with $p=2$). Thus $[\mathbb{Q}(\sqrt[4]{2}):\mathbb{Q}] = \deg(m) = 4$.
A basis is $\{1, \sqrt[4]{2}, \sqrt{2}, \sqrt[4]{8}\}$.

7. If $[K:F] = 2$, prove that $K = F(\alpha)$ for some $\alpha \in K$ with $\alpha^2 \in F$.

Solution. Let $\alpha \in K \setminus F$. Then $[F(\alpha):F] \geq 2$. Since $[K:F] = 2$ and $F \subsetneq F(\alpha) \subseteq K$, we must have $F(\alpha) = K$ and $[F(\alpha):F] = 2$.
The minimal polynomial of α over F has degree 2, so $m(x) = x^2 + bx + c$ for some $b, c \in F$. Completing the square: $\beta = \alpha + b/2$ satisfies $\beta^2 = \alpha^2 + b\alpha + b^2/4 = -c + b^2/4 \in F$.
Since $\mathbb{Q}(\alpha) = \mathbb{Q}(\beta)$, we can take this β as our element with $\beta^2 \in F$.

8. Prove that if $[\mathbb{Q}(\alpha):\mathbb{Q}]$ is odd, then $\mathbb{Q}(\alpha) = \mathbb{Q}(\alpha^2)$.

Solution. Clearly $\mathbb{Q}(\alpha^2) \subseteq \mathbb{Q}(\alpha)$. For the reverse inclusion, note that α satisfies $x^2 - \alpha^2 = 0$ over $\mathbb{Q}(\alpha^2)$, so $[\mathbb{Q}(\alpha):\mathbb{Q}(\alpha^2)] \leq 2$. By the tower law: $[\mathbb{Q}(\alpha):\mathbb{Q}] = [\mathbb{Q}(\alpha):\mathbb{Q}(\alpha^2)]\cdot[\mathbb{Q}(\alpha^2):\mathbb{Q}]$. Since $[\mathbb{Q}(\alpha):\mathbb{Q}]$ is odd and $[\mathbb{Q}(\alpha):\mathbb{Q}(\alpha^2)] \leq 2$, we must have $[\mathbb{Q}(\alpha):\mathbb{Q}(\alpha^2)] = 1$. Therefore $\mathbb{Q}(\alpha) = \mathbb{Q}(\alpha^2)$.

9. Find the minimal polynomial of $\sqrt{2}+i$ over $\mathbb{Q}$.

> ***Solution.*** Let $\alpha = \sqrt{2}+i$. Then $\alpha - i = \sqrt{2}$, so $(\alpha - i)^2 = 2$, giving $\alpha^2 - 2i\alpha + i^2 = 2$, thus $\alpha^2 - 1 = 2i\alpha$. Squaring: $(\alpha^2-1)^2 = -4\alpha^2$, so $\alpha^4 - 2\alpha^2 + 1 = -4\alpha^2$, giving $\alpha^4 + 2\alpha^2 + 1 = 0$, which factors as $(\alpha^2+1)^2 = 0$. However, this means $\alpha^2 = -1$. Let's verify: $(\sqrt{2}+i)^2 = 2 + 2i\sqrt{2} - 1 = 1 + 2i\sqrt{2} \neq -1$. Let me recalculate: $\alpha^2 - 1 = 2i\alpha$ gives $(\alpha^2-1)^2 = -4\alpha^2$, so $\alpha^4 - 2\alpha^2 + 1 + 4\alpha^2 = 0$, thus $m(x) = x^4 + 2x^2 + 1 = (x^2+1)^2$. But this is reducible. The correct approach: $\alpha - \sqrt{2} = i$, so $(\alpha - \sqrt{2})^2 = -1$, giving $\alpha^2 - 2\sqrt{2}\alpha + 2 + 1 = 0$. Then $\alpha^2 + 3 = 2\sqrt{2}\alpha$. Squaring: $\alpha^4 + 6\alpha^2 + 9 = 8\alpha^2$, so $m(x) = x^4 - 2x^2 + 9$.

10. Prove that $[\mathbb{Q}(\sqrt{2},\sqrt{3}) : \mathbb{Q}(\sqrt{2}+\sqrt{3})] = 2$.

> ***Solution.*** From Problem 16 of Chapter 11, $\mathbb{Q}(\sqrt{2}+\sqrt{3}) = \mathbb{Q}(\sqrt{2},\sqrt{3})$ with $[\mathbb{Q}(\sqrt{2},\sqrt{3}) : \mathbb{Q}] = 4$. Wait, that would make the ratio 1, not 2. Let me reconsider. Actually, if $\mathbb{Q}(\sqrt{2}+\sqrt{3}) = \mathbb{Q}(\sqrt{2},\sqrt{3})$, then the ratio is 1. The problem statement may be asking for something else. If we consider $\mathbb{Q}(\sqrt{6}) \subseteq \mathbb{Q}(\sqrt{2},\sqrt{3})$, then $[\mathbb{Q}(\sqrt{2},\sqrt{3}) : \mathbb{Q}(\sqrt{6})] = 2$ (can be verified).

11. Show that the minimal polynomial of $\cos(2\pi/5)$ over $\mathbb{Q}$ has degree 2.

> ***Solution.*** Let $\zeta = e^{2\pi i/5}$, a primitive 5th root of unity. Then $\zeta^5 = 1$ and ζ satisfies the cyclotomic polynomial $\Phi_5(x) = x^4 + x^3 + x^2 + x + 1$ (irreducible over $\mathbb{Q}$), so $[\mathbb{Q}(\zeta) : \mathbb{Q}] = 4$. Now $\cos(2\pi/5) = \frac{\zeta + \zeta^{-1}}{2} = \frac{\zeta + \zeta^4}{2} \in \mathbb{R}$. Since $[\mathbb{Q}(\zeta) : \mathbb{Q}] = 4$ and $[\mathbb{Q}(\zeta) : \mathbb{Q}(\cos(2\pi/5))] = 2$ (because $\mathbb{Q}(\zeta)$ is generated over $\mathbb{Q}(\cos(2\pi/5))$ by $i\sin(2\pi/5)$, which satisfies a degree-2 equation), we have $[\mathbb{Q}(\cos(2\pi/5)) : \mathbb{Q}] = 2$.

12. If $[K : F] = n$, prove that every element of K is algebraic over F of degree at most n.

> ***Solution.*** Let $\alpha \in K$. The elements $1, \alpha, \alpha^2, \ldots, \alpha^n$ are $n+1$ elements in the n-dimensional vector space K over F, hence linearly dependent. Thus there exist $a_0, \ldots, a_n \in F$ (not all zero) with $\sum_{i=0}^{n} a_i \alpha^i = 0$. Therefore α is algebraic over F. The degree of α is $[F(\alpha) : F]$, which divides $[K : F] = n$ by the tower law, so the degree is at most n.

13. Determine whether $\sqrt{2} + \sqrt[3]{2}$ is algebraic or transcendental over $\mathbb{Q}$.

> ***Solution.*** Let $\alpha = \sqrt{2} + \sqrt[3]{2}$. Since $\alpha = \sqrt{2} + \sqrt[3]{2} \in \mathbb{Q}(\sqrt{2}, \sqrt[3]{2})$ and this is a finite extension of $\mathbb{Q}$ (degree ≤ 6, in fact exactly 6), every element of a finite extension is algebraic. Therefore α is algebraic over $\mathbb{Q}$.

14. Prove that $\mathbb{Q}(\sqrt{2})(\sqrt{3}) = \mathbb{Q}(\sqrt{3})(\sqrt{2})$.

> ***Solution.*** Both equal $\mathbb{Q}(\sqrt{2}, \sqrt{3})$, the smallest field containing $\mathbb{Q}$, $\sqrt{2}$, and $\sqrt{3}$. The notation $F(\alpha)(\beta)$ means $(F(\alpha))(\beta) = F(\alpha, \beta)$, which is symmetric in α and β.

15. If α has minimal polynomial $m(x)$ of degree n over F, show that $\{1, \alpha, \alpha^2, \ldots, \alpha^{n-1}\}$ is a basis for $F(\alpha)$ over F.

> ***Solution.*** **Spanning:** Any $\beta \in F(\alpha) = F[\alpha]$ is a polynomial in α. For $k \geq n$, we can reduce α^k modulo $m(\alpha) = 0$ to express it as a linear combination of $1, \alpha, \ldots, \alpha^{n-1}$.
> **Linear independence:** If $\sum_{i=0}^{n-1} c_i \alpha^i = 0$ with $c_i \in F$, then α is a root of $f(x) = \sum_{i=0}^{n-1} c_i x^i$. If some $c_i \neq 0$, then $\deg(f) < n = \deg(m)$, contradicting that m is the minimal polynomial. Thus all $c_i = 0$.

16. Find $[\mathbb{Q}(\sqrt[3]{2}, \omega) : \mathbb{Q}(\omega)]$ where $\omega = e^{2\pi i/3}$.

> ***Solution.*** The element $\sqrt[3]{2}$ satisfies $x^3 - 2 = 0$. Over $\mathbb{Q}(\omega)$, the polynomial $x^3 - 2$ factors (since $\omega\sqrt[3]{2}$ and $\omega^2\sqrt[3]{2}$ are also roots), but it's irreducible if $\sqrt[3]{2} \notin \mathbb{Q}(\omega)$. Since $\mathbb{Q}(\omega) \subset \mathbb{C}$ and $\sqrt[3]{2} \in \mathbb{R}$ while $\mathbb{Q}(\omega) \cap \mathbb{R} = \mathbb{Q}$ (as $[\mathbb{Q}(\omega) : \mathbb{Q}] = 2$ and $\omega \notin \mathbb{R}$), we have $\sqrt[3]{2} \notin \mathbb{Q}(\omega)$. The minimal polynomial of $\sqrt[3]{2}$ over $\mathbb{Q}(\omega)$ has degree dividing 3. Since $\sqrt[3]{2} \notin \mathbb{Q}(\omega)$, the degree is 3. Thus $[\mathbb{Q}(\sqrt[3]{2}, \omega) : \mathbb{Q}(\omega)] = 3$.

17. Prove that if $[K : F]$ and $[\ell : F]$ are coprime, then $K \cap \ell = F$.

> ***Solution.*** Let $E = K \cap \ell$. Then $F \subseteq E \subseteq K$ and $F \subseteq E \subseteq \ell$, so $[K : F] = [K : E][E : F]$ and $[\ell : F] = [\ell : E][E : F]$. Thus $[E : F]$ divides both $[K : F]$ and $[\ell : F]$. Since these are coprime, $[E : F] = 1$, so $E = F$.

18. Show that $[\mathbb{C} : \mathbb{Q}] = \infty$.

> ***Solution.*** If $[\mathbb{C} : \mathbb{Q}] = n < \infty$, then every element of $\mathbb{C}$ would be algebraic over $\mathbb{Q}$ of degree at most n. But π is transcendental over $\mathbb{Q}$, so $\pi \in \mathbb{C}$ is not algebraic. Contradiction. Therefore $[\mathbb{C} : \mathbb{Q}] = \infty$.

19. If $K = F(\alpha, \beta)$ with $[\mathbb{F}(\alpha) : F] = m$ and $[F(\beta) : F] = n$ where $\gcd(m, n) = 1$, prove that $[K : F] = mn$.

> ***Solution.*** We have $F \subseteq F(\alpha) \subseteq F(\alpha, \beta) = K$ and $F \subseteq F(\beta) \subseteq K$. By the tower law:
> $[K : F] = [K : F(\alpha)][F(\alpha) : F] = [K : F(\alpha)] \cdot m$. So $m \mid [K : F]$. Similarly, $n \mid [K : F]$. Since $\gcd(m, n) = 1$, we have $mn \mid [K : F]$. Also, $[K : F] = [K : F(\beta)][F(\beta) : F] \leq [F(\alpha, \beta) : F(\beta)] \cdot n$. We have $F(\beta) \subseteq F(\alpha, \beta)$ and α satisfies a polynomial of degree m over F, hence over $F(\beta)$, so $[F(\alpha, \beta) : F(\beta)] \leq m$. Thus $[K : F] \leq mn$. Combined with $mn \mid [K : F]$ and both finite, we get $[K : F] = mn$.

20. Prove that $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{6}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

> ***Solution.*** Clearly $\mathbb{Q}(\sqrt{2},\sqrt{3}) \subseteq \mathbb{Q}(\sqrt{2},\sqrt{3},\sqrt{6})$. For the reverse, note that $\sqrt{6} = \sqrt{2}\cdot\sqrt{3} \in \mathbb{Q}(\sqrt{2},\sqrt{3})$. Thus $\mathbb{Q}(\sqrt{2},\sqrt{3},\sqrt{6}) \subseteq \mathbb{Q}(\sqrt{2},\sqrt{3})$. Therefore they are equal.

End of Chapter 12

Chapter 13: Algebraic Extensions

❦

Algebraic extensions represent the natural domain where polynomial equations find their solutions. Understanding when all elements of an extension are algebraic, exploring algebraically closed fields, and distinguishing separable from inseparable phenomena reveals the deep structure underlying field theory and prepares the groundwork for Galois theory.

13.1 Algebraic Extensions

> **Definition**
>
> A field extension K/F is **algebraic** if every element of K is algebraic over F That is, for each $\alpha \in K$, there exists a nonzero polynomial $f \in F[x]$ with $f(\alpha) = 0$.

> **Example**
>
> $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$ is algebraic since $\sqrt{2}$ and $\sqrt{3}$ are algebraic over $\mathbb{Q}$, and sums, products, and quotients of algebraic elements remain algebraic.

Example

$\mathbb{C}/\mathbb{R}$ is algebraic: every $z = a + bi$ satisfies $(x - z)(x - \overline{z}) = x^2 - 2ax + (a^2 + b^2) \in \mathbb{R}[x]$.

Example

$\mathbb{Q}(x)/\mathbb{Q}$ is not algebraic: x is transcendental over $\mathbb{Q}$.

Theorem

Every finite extension is algebraic.

> ***Proof.*** Let K/F be finite with $[K : F] = n$. For any $\alpha \in K$, the elements $1, \alpha, \alpha^2, \ldots, \alpha^n$ are $n + 1$ elements in an n-dimensional vector space over F, hence linearly dependent. Thus there exist $a_0, \ldots, a_n \in F$ (not all zero) with $\sum_{i=0}^{n} a_i \alpha^i = 0$, so α is algebraic over F. ∎

If $[K : F] < \infty$, then $K = F(\alpha_1, \ldots, \alpha_n)$ where each α_i is algebraic over F.

The converse is false: $\mathbb{Q}$ adjoined with all algebraic numbers forms an infinite extension that is algebraic but not finite.

13.2 Properties of Algebraic Extensions

Theorem

(Tower Property) If $F \subseteq E \subseteq K$ and E/F and K/E are both algebraic, then K/F is algebraic.

Proof. Let $\alpha \in K$. Since K/E is algebraic, α satisfies $\sum_{i=0}^{n} e_i \alpha^i = 0$ with $e_i \in E$. Let $L = F(e_0, \ldots, e_n)$. Since E/F is algebraic, each e_i is algebraic over F, so L/F is finite (repeated application of the tower law). Since α satisfies a polynomial with coefficients in L, we have $[L(\alpha) : L] < \infty$. Thus $[L(\alpha) : F] = [L(\alpha) : L] \cdot [L : F] < \infty$. Since $F \subseteq F(\alpha) \subseteq L(\alpha)$ and $[L(\alpha) : F] < \infty$, we have $[F(\alpha) : F] < \infty$, so α is algebraic over F. ∎

Theorem

If α, β are algebraic over F, then $\alpha + \beta$, $\alpha\beta$, and α/β (if $\beta \neq 0$) are algebraic over F.

Proof. Since $[\mathbb{F}(\alpha) : F] < \infty$ and $[F(\alpha, \beta) : F(\alpha)] < \infty$, the tower law gives $[F(\alpha, \beta) : F] < \infty$. Since $\alpha \pm \beta, \alpha\beta, \alpha/\beta \in F(\alpha, \beta)$ and $F(\alpha, \beta)/F$ is finite (hence algebraic), these elements are algebraic over F. ∎

The set of all algebraic elements in a field extension forms a subfield.

Definition

The **algebraic closure** of F in K, denoted $\overline{F}^K$, is the set of all elements of K that are algebraic over F. This is a subfield of K containing F.

Example

The algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ is $\overline{\mathbb{Q}} = \{\alpha \in \mathbb{C} : \alpha \text{ is algebraic over } \mathbb{Q}\}$, the field of algebraic numbers.

13.3 Algebraically Closed Fields

Definition

A field F is **algebraically closed** if every nonconstant polynomial in $F[x]$ has a root in F. Equivalently:

- Every polynomial in $F[x]$ splits completely into linear factors over F
- Every irreducible polynomial in $F[x]$ has degree 1
- F has no proper algebraic extensions

Example

(Fundamental Theorem of Algebra) $\mathbb{C}$ is algebraically closed. Every polynomial in $\mathbb{C}[x]$ of degree $n \geq 1$ has exactly n roots (counting multiplicity) in $\mathbb{C}$.

Example

$\mathbb{R}$ is not algebraically closed: $x^2 + 1$ has no roots in $\mathbb{R}$.

Example

$\mathbb{Q}$ is not algebraically closed: $x^2 - 2$ has no roots in $\mathbb{Q}$.

Theorem

The following are equivalent for a field F:

1. F is algebraically closed
2. Every nonconstant $f \in F[x]$ has a root in F
3. Every nonconstant $f \in F[x]$ splits completely into linear factors
4. The only irreducible polynomials in $F[x]$ are linear
5. If K/F is algebraic, then $K = F$

> ***Proof.*** $(1 \Rightarrow 2)$: Definition. $(2 \Rightarrow 3)$: By induction on degree. $(3 \Rightarrow 4)$: Clear. $(4 \Rightarrow 1)$: Every nonconstant polynomial is a product of linear factors. $(1 \Rightarrow 5)$: If $\alpha \in K$, then α is algebraic, so its minimal polynomial over F is linear (by (4)), thus $\alpha \in F$. $(5 \Rightarrow 1)$: If $f \in F[x]$ is nonconstant, adjoin a root in some extension; since this extension must equal F, the root is in F. ∎

13.4 Algebraic Closure

> **Definition**
>
> An **algebraic closure** of F is an algebraic extension $\overline{F}/F$ such that $\overline{F}$ is algebraically closed.

> **Theorem**
>
> **(Existence and Uniqueness)** Every field F has an algebraic closure $\overline{F}$, unique up to isomorphism fixing F.

The existence proof uses Zorn's lemma and transfinite methods. Uniqueness follows from extending isomorphisms along algebraic extensions.

> **Example**
>
> The algebraic closure of $\mathbb{Q}$ is $\overline{\mathbb{Q}}$, the field of all algebraic numbers. We have $\overline{\mathbb{Q}} \subsetneq \mathbb{C}$ since $\pi, e \in \mathbb{C} \setminus \overline{\mathbb{Q}}$.

Example

$\overline{\mathbb{R}} = \mathbb{C}$. Since $\mathbb{C}$ is algebraically closed and $[\mathbb{C} : \mathbb{R}] = 2 < \infty$, $\mathbb{C}$ is the algebraic closure of $\mathbb{R}$.

Example

The algebraic closure $\overline{\mathbb{F}_p}$ of the finite field $\mathbb{F}_p$ is the union $\bigcup_{n=1}^{\infty} \mathbb{F}_{p^n}$.

Proposition

If F is algebraically closed and K/F is algebraic, then $K = F$.

Theorem

$[\overline{F} : F] = \infty$ for any field F that is not algebraically closed.

> ***Proof.*** If $[\overline{F} : F] < \infty$, then $\overline{F}/F$ would be finite. But every finite extension K of an algebraically closed field equals the field itself, contradiction. ∎

13.5 Separable and Inseparable Extensions

Definition

A polynomial $f \in F[x]$ is **separable** if its irreducible factors have no repeated roots in a splitting field. An element α is **separable over** F if its minimal polynomial is separable. An extension K/F is **separable** if every element of K is separable over F.

Theorem

A polynomial $f \in F[x]$ is separable if and only if $\gcd(f, f') = 1$, where f' is the formal derivative.

> ***Proof.*** If f has a repeated root α, then $f(x) = (x - \alpha)^m g(x)$ with $m \geq 2$. Then $f'(x) = m(x - \alpha)^{m-1} g(x) + (x - \alpha)^m g'(x)$, so $(x - \alpha)$ divides both f and f', thus $\gcd(f, f') \neq 1$. Conversely, if $\gcd(f, f') = d \neq 1$, then d has a root α in some extension, which must also be a root of f and f', implying f has α as a repeated root. ∎

Theorem

In characteristic zero, every polynomial is separable.

> ***Proof.*** If f is irreducible of degree $n \geq 1$ in characteristic zero, then $f' \neq 0$ (since the leading coefficient of f' is a nonzero multiple of n). Since $\deg(f') < \deg(f)$ and f is irreducible, $\gcd(f, f') = 1$. ∎

Every algebraic extension of a field of characteristic zero is separable.

> **Example**
>
> Over $\mathbb{F}_p(t)$ where t is an indeterminate, the polynomial $f(x) = x^p - t$ is irreducible but inseparable. In the splitting field, $f(x) = (x - \alpha)^p$ where $\alpha^p = t$, so α is a repeated root.

> **Definition**
>
> A field F is **perfect** if every algebraic extension of F is separable. Equivalently:
>
> - $\mathrm{char}(F) = 0$, or
> - $\mathrm{char}(F) = p > 0$ and every element of F is a p-th power

> **Theorem**
>
> Every finite field is perfect.

> ***Proof.*** If $|F| = q = p^n$, the Frobenius map $\varphi(x) = x^p$ is an automorphism of F (injective endomorphism of a finite set), hence surjective. Thus every element is a p-th power. ∎

Greek Meditation: Closure and Completion

In Aristotelian philosophy, every substance strives toward its telos—its natural end and perfection. The algebraic closure of a field represents this mathematical telos: the completion toward which every field naturally extends. Just as the acorn contains the potential oak, every field F contains implicitly its algebraic closure $\bar{F}$, where all polynomial equations find their roots. This closure is not an external addition but an internal completion, the fulfillment of what was always latent. The algebraic closure reveals that incompleteness is not permanent but provisional—every field can reach its perfect form where all algebraic questions receive answers.

Zen Meditation: Separability

A Zen koan asks: What was your original face before your parents were born? In characteristic zero, every element shows its original face clearly—roots are always distinct, separable, individuated. But in characteristic p, multiplicity can arise; roots may merge into one. Separability is the mathematical expression of distinctness-in-unity: one polynomial, many roots, each preserving its identity. Inseparability, conversely, shows how the many can collapse into the one. Like the ten thousand things arising from and returning to the Tao, separable extensions maintain distinction while inseparable extensions reveal the underlying unity. To understand both is to transcend the duality of one and many.

Practice Problems

1. Prove that $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$ is algebraic over $\mathbb{Q}$.

> ***Solution.*** Since $[\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}) : \mathbb{Q}] = 8 < \infty$ (from earlier chapter), this is a finite extension, hence algebraic.

2. Show that if $\alpha^3 = 2$ and $\beta^2 = \alpha$, then β is algebraic over $\mathbb{Q}$ of degree dividing 6.

> ***Solution.*** $\beta^2 = \alpha$ and $\alpha^3 = 2$ give $(\beta^2)^3 = 2$, so $\beta^6 = 2$. Thus β satisfies $x^6 - 2 = 0$ over $\mathbb{Q}$. The minimal polynomial divides $x^6 - 2$, so $[\mathbb{Q}(\beta) : \mathbb{Q}] \mid 6$.

3. Prove that the set of algebraic numbers $\overline{\mathbb{Q}}$ is countable.

> ***Solution.*** For each n, there are countably many polynomials of degree n with integer coefficients (in bijection with $\mathbb{Z}^{n+1}$), and each has finitely many roots. Thus $\overline{\mathbb{Q}}$ is a countable union of finite sets, hence countable.

4. Is $\mathbb{Q}[x]$ algebraically closed?

> ***Solution.*** No. The polynomial $x - t \in \mathbb{Q}(x)[y]$ has no root in $\mathbb{Q}(x)$ (where y is a different indeterminate).

5. Show that if F is algebraically closed, then so is $F(x)$? (True or false)

> ***Solution.*** False. Even if F is algebraically closed, $F(x)$ is not. For example, $\mathbb{C}$ is algebraically closed, but $\mathbb{C}(x)$ is not—the polynomial $y^2 - x \in \mathbb{C}(x)[y]$ has no root in $\mathbb{C}(x)$.

6. Determine whether $x^{p^n} - x$ is separable over $\mathbb{F}_p$.

> ***Solution.*** The derivative is $(x^{p^n} - x)' = p^n x^{p^n-1} - 1 = -1$ (in characteristic p). Since $\gcd(x^{p^n} - x, -1) = 1$, the polynomial is separable.

7. Prove that every extension of a perfect field is perfect.

> ***Solution.*** Let F be perfect and K/F an extension. If $\mathrm{char}(F) = 0$, then $\mathrm{char}(K) = 0$, so K is perfect. If $\mathrm{char}(F) = p$ and every element of F is a p-th power, then for $\alpha \in K$, if α is algebraic over F, its minimal polynomial is separable (since F is perfect), so every algebraic extension of K is separable.

8. Show that $\overline{\mathbb{Q}} \cap \mathbb{R} \neq \overline{\mathbb{Q}}$.

> ***Solution.*** Since $i \in \overline{\mathbb{Q}}$ (as $i^2 + 1 = 0$) but $i \notin \mathbb{R}$, we have $i \notin \overline{\mathbb{Q}} \cap \mathbb{R}$, so $\overline{\mathbb{Q}} \cap \mathbb{R} \subsetneq \overline{\mathbb{Q}}$.

9. Prove $\mathrm{char}(\overline{F}) = \mathrm{char}(F)$.

> ***Solution.*** Since $F \subseteq \overline{F}$, the characteristic is preserved under field extensions.

10. Determine whether $\mathbb{F}_p(t)$ is perfect, where t is transcendental.

> ***Solution.*** No. t is not a p-th power in $\mathbb{F}_p(t)$ (if $t = (f/g)^p$, then $tg^p = f^p$, contradiction). Thus $\mathbb{F}_p(t)$ is not perfect.

11. Show that if $[K : F] = 2$, then K/F is algebraic.

> ***Solution.*** Finite extensions are algebraic (Theorem in §13.1).

12. Prove that an algebraically closed field is infinite.

> ***Solution.*** If $F = \{a_1, \ldots, a_n\}$ is finite, consider $f(x) = (x - a_1)(x - a_2) \cdots (x - a_n) + 1$. This polynomial has no roots in F, contradicting that F is algebraically closed.

13. Is $\sqrt{2} + \sqrt{3}$ separable over $\mathbb{Q}$?

> ***Solution.*** Yes. In characteristic zero, all algebraic elements are separable.

14. Show that $x^4 + 1$ is separable over $\mathbb{Q}$.

> ***Solution.*** $(x^4 + 1)' = 4x^3$. Since $\gcd(x^4 + 1, 4x^3) = 1$ (no common roots), $x^4 + 1$ is separable.

15. Determine $[\overline{\mathbb{F}_p} : \mathbb{F}_p]$.

> ***Solution.*** $[\overline{\mathbb{F}_p} : \mathbb{F}_p] = \infty$ since $\mathbb{F}_p$ is not algebraically closed and the algebraic closure has infinite degree over any non-algebraically-closed field.

16. Prove that if K is algebraically closed and K/F is algebraic, then $K = \overline{F}$.

> ***Solution.*** K is an algebraic extension of F that is algebraically closed, which is the definition of an algebraic closure.

17. Show that separability is preserved under finite extensions.

> ***Solution.*** If K/F is separable and L/K is separable, then L/F is separable by the tower property for separable extensions.

18. Is $\mathbb{Q}(\pi)$ algebraic over $\mathbb{Q}$?

> ***Solution.*** No. Since π is transcendental over $\mathbb{Q}$, the extension $\mathbb{Q}(\pi)/\mathbb{Q}$ is not algebraic.

19. Prove that $\mathbb{C}$ is the algebraic closure of $\mathbb{R}$.

> ***Solution.*** $\mathbb{C}$ is algebraically closed (Fundamental Theorem of Algebra) and $[\mathbb{C} : \mathbb{R}] = 2 < \infty$, so $\mathbb{C}/\mathbb{R}$ is algebraic. Hence $\mathbb{C} = \overline{\mathbb{R}}$.

20. Show that the algebraic closure of $\mathbb{F}_p$ in $\mathbb{F}_{p^{n!}}$ contains $\mathbb{F}_{p^k}$ for all $k \leq n$.

> ***Solution.*** For $k \leq n$, we have $k \mid n!$, so $\mathbb{F}_{p^k} \subseteq \mathbb{F}_{p^{n!}}$ (as subfields correspond to divisors of the degree). Since each $\mathbb{F}_{p^k}/\mathbb{F}_p$ is algebraic (finite), every element is algebraic over $\mathbb{F}_p$.

✦

End of Chapter 13

Chapter 14: Splitting Fields and Galois Extensions

"In every polynomial lies the seed of its own perfection, awaiting only the proper field in which to bloom."

The concept of a splitting field represents one of the most elegant ideas in field theory. Given a polynomial over a field, we seek the smallest extension in which the polynomial factors completely into linear terms. This chapter explores the existence, uniqueness, and fundamental properties of splitting fields, establishing the groundwork for Galois theory.

14.1 Splitting Fields: Definition and Examples

We begin with the fundamental definition that captures when a polynomial "splits completely" over a field.

> **Definition**
>
> **Definition 14.1.1 (Polynomial Splits Completely).** Let F be a field and $f(x) \in F[x]$ a polynomial of degree $n \geq 1$. We say that $f(x)$ *splits completely* over F if
>
> $$f(x) = a(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$$
>
> where $a \in F^*$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$.

Remark 14.1.2. Note that the roots α_i need not be distinct. The polynomial $f(x) = (x - 2)^3$ splits completely over $\mathbb{Q}$, with root $\alpha = 2$ having multiplicity 3.

Definition

Definition 14.1.3 (Splitting Field). Let F be a field and $f(x) \in F[x]$ a nonconstant polynomial. An extension K of F is a *splitting field* for $f(x)$ over F if:

1. $f(x)$ splits completely over K, and
2. $K = F(\alpha_1, \alpha_2, \ldots, \alpha_n)$ where $\alpha_1, \ldots, \alpha_n$ are all the roots of $f(x)$ in K.

The second condition ensures that K is the smallest field containing F in which $f(x)$ splits completely. This minimality is crucial for uniqueness.

14.1.1 Motivating Examples

Example

Example 14.1.4 (Splitting Field of $x^2 - 2$ over $\mathbb{Q}$). Consider $f(x) = x^2 - 2 \in \mathbb{Q}[x]$. The roots of $f(x)$ are $\pm\sqrt{2}$. We have

$$f(x) = (x - \sqrt{2})(x + \sqrt{2}).$$

The field $K = \mathbb{Q}(\sqrt{2})$ is a splitting field for $f(x)$ over $\mathbb{Q}$ because:

1. $f(x)$ splits completely in K: both roots $\pm\sqrt{2}$ lie in K.
2. $K = \mathbb{Q}(\sqrt{2}) = \mathbb{Q}(\sqrt{2}, -\sqrt{2})$ since $-\sqrt{2} = -1 \cdot \sqrt{2} \in \mathbb{Q}(\sqrt{2})$.

Note that $[K : \mathbb{Q}] = 2$ since $\{1, \sqrt{2}\}$ is a basis for K over $\mathbb{Q}$.

Example

Example 14.1.5 (Splitting Field of $x^3 - 2$ over $\mathbb{Q}$). Let $f(x) = x^3 - 2 \in \mathbb{Q}[x]$. The roots are:

$$\alpha_1 = \sqrt[3]{2} \in \mathbb{R}$$

$$\alpha_2 = \sqrt[3]{2} \cdot \omega = \sqrt[3]{2} \cdot e^{2\pi i/3}$$

$$\alpha_3 = \sqrt[3]{2} \cdot \omega^2 = \sqrt[3]{2} \cdot e^{4\pi i/3}$$

where $\omega = e^{2\pi i/3} = \frac{-1+i\sqrt{3}}{2}$ is a primitive cube root of unity.

The splitting field is $K = \mathbb{Q}(\sqrt[3]{2}, \omega)$. Note that:

- $\mathbb{Q}(\sqrt[3]{2}) \subsetneq K$ since $\sqrt[3]{2} \in \mathbb{R}$ but $\omega \notin \mathbb{R}$.
- $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$ (minimal polynomial: $x^3 - 2$).
- $[\mathbb{Q}(\omega) : \mathbb{Q}] = 2$ (minimal polynomial: $x^2 + x + 1$).
- $[K : \mathbb{Q}] = 6$ (we'll prove this using degree formulas).

A basis for K over $\mathbb{Q}$ is:

$$\{1, \omega, \sqrt[3]{2}, \sqrt[3]{2}\omega, \sqrt[3]{4}, \sqrt[3]{4}\omega\}.$$

Example

Example 14.1.6 (Splitting Field of $x^4 - 1$ over $\mathbb{Q}$). Consider $f(x) = x^4 - 1 \in \mathbb{Q}[x]$. The roots are the fourth roots of unity:

$$1, -1, i, -i.$$

We can factor:

$$x^4 - 1 = (x - 1)(x + 1)(x - i)(x + i).$$

The splitting field is $K = \mathbb{Q}(1, -1, i, -i) = \mathbb{Q}(i)$ since $1, -1 \in \mathbb{Q}$ already. Note that $[K : \mathbb{Q}] = 2$ with basis $\{1, i\}$.

Example

Example 14.1.7 (Splitting Field of $x^p - x$ over $\mathbb{F}_p$). Let p be prime and consider $f(x) = x^p - x \in \mathbb{F}_p[x]$. By Fermat's Little Theorem, every element $a \in \mathbb{F}_p$ satisfies $a^p = a$, so $a^p - a = 0$. Thus all p elements of $\mathbb{F}_p$ are roots of $f(x)$:

$$x^p - x = \prod_{a \in \mathbb{F}_p} (x - a).$$

Therefore, $\mathbb{F}_p$ itself is the splitting field of $x^p - x$ over $\mathbb{F}_p$.

This observation generalizes: the finite field $\mathbb{F}_{p^n}$ is the splitting field of $x^{p^n} - x$ over $\mathbb{F}_p$.

Example

Example 14.1.8 (Non-Example: $\mathbb{Q}(\sqrt{2})$ and $x^4 - 4$). Consider $f(x) = x^4 - 4 = (x^2 - 2)(x^2 + 2)$. The roots are:

$$\pm\sqrt{2}, \pm i\sqrt{2}.$$

The field $\mathbb{Q}(\sqrt{2})$ contains two roots ($\pm\sqrt{2}$) but not all roots. Thus $\mathbb{Q}(\sqrt{2})$ is not a splitting field for $f(x)$ over $\mathbb{Q}$.

The splitting field is $K = \mathbb{Q}(\sqrt{2}, i\sqrt{2}) = \mathbb{Q}(\sqrt{2}, i)$ with $[K : \mathbb{Q}] = 4$.

14.1.2 More Sophisticated Examples

Example

Example 14.1.9 (Splitting Field of $x^4 + x^2 + 1$ over $\mathbb{Q}$). Consider $f(x) = x^4 + x^2 + 1$. We can factor:

$$x^4 + x^2 + 1 = (x^2 - x + 1)(x^2 + x + 1).$$

The roots of $x^2 - x + 1 = 0$ are $\omega = e^{2\pi i/3}$ and ω^2. The roots of $x^2 + x + 1 = 0$ are $-\omega^2$ and $-\omega$.

Actually, $x^4 + x^2 + 1 = \frac{x^6 - 1}{x^2 - 1}$, so the roots are primitive 6th roots of unity:

$$e^{2\pi i/6}, e^{4\pi i/6}, e^{8\pi i/6}, e^{10\pi i/6}.$$

Simplifying: $e^{\pi i/3}, e^{2\pi i/3}, e^{4\pi i/3}, e^{5\pi i/3}$. Let $\zeta = e^{\pi i/3}$. Then all roots are powers of ζ, so the splitting field is $K = \mathbb{Q}(\zeta)$. Since the minimal polynomial of ζ over $\mathbb{Q}$ is $x^2 - x + 1$ (as $\zeta = \frac{1 + i\sqrt{3}}{2}$), we have $[K : \mathbb{Q}] = 2$.

Example

Example 14.1.10 (Splitting Field of $(x^2-2)(x^2-3)$ over $\mathbb{Q}$). Let $f(x) = (x^2-2)(x^2-3) = x^4 - 5x^2 + 6$. The roots are:

$$\pm\sqrt{2}, \pm\sqrt{3}.$$

The splitting field is $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

To find $[K : \mathbb{Q}]$, we build a tower:

$$\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt{2}, \sqrt{3}).$$

We have $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$. For the second extension:

- If $\sqrt{3} \in \mathbb{Q}(\sqrt{2})$, then $\sqrt{3} = a + b\sqrt{2}$ for some $a, b \in \mathbb{Q}$.
- Squaring: $3 = a^2 + 2b^2 + 2ab\sqrt{2}$.
- This requires $2ab = 0$ and $a^2 + 2b^2 = 3$.
- If $b = 0$: $a^2 = 3$, so $a = \pm\sqrt{3} \notin \mathbb{Q}$. Contradiction.
- If $a = 0$: $2b^2 = 3$, so $b = \pm\sqrt{3/2} \notin \mathbb{Q}$. Contradiction.

Therefore $\sqrt{3} \notin \mathbb{Q}(\sqrt{2})$, and the minimal polynomial of $\sqrt{3}$ over $\mathbb{Q}(\sqrt{2})$ is $x^2 - 3$, giving $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{2})] = 2$.

Thus $[K : \mathbb{Q}] = [\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{2})] \cdot [\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2 \cdot 2 = 4$.

A basis for K over $\mathbb{Q}$ is $\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$.

14.2 Existence of Splitting Fields

Having seen examples, we now prove that splitting fields always exist. The proof is constructive, building the field by successively adjoining roots.

Theorem

Theorem 14.2.1 (Existence of Splitting Fields). Let F be a field and $f(x) \in F[x]$ a nonconstant polynomial. Then there exists a splitting field K for $f(x)$ over F. Moreover, $[K : F] \leq n!$ where $n = \deg(f)$.

Proof. We proceed by induction on $n = \deg(f)$.

Base case: $n = 1$. Then $f(x) = a(x - \alpha)$ for some $a \in F^*$ and $\alpha \in F$. The polynomial already splits completely over F, so $K = F$ is the splitting field with $[K : F] = 1 \leq 1!$.

Inductive step: Assume $n \geq 2$ and the theorem holds for all polynomials of degree less than n. Let $p(x)$ be an irreducible factor of $f(x)$ in $F[x]$ with $\deg(p) \geq 1$. Consider the extension:

$$F_1 = F[x]/(p(x)).$$

Let $\alpha \in F_1$ be the image of x under the quotient map. Then:

- F_1 is a field extension of F.
- $p(\alpha) = 0$ in F_1.
- $[F_1 : F] = \deg(p) \leq n$.

Since α is a root of $p(x)$ and $p(x) \mid f(x)$, we have $f(\alpha) = 0$ in F_1. Therefore, in $F_1[x]$:

$$f(x) = (x - \alpha)g(x)$$

where $\deg(g) = n - 1$.

By the inductive hypothesis, there exists a splitting field K for $g(x)$ over F_1 with:

$$[K : F_1] \leq (n - 1)!.$$

In $K[x]$, the polynomial $g(x)$ splits completely as:

$$g(x) = c(x - \alpha_2)(x - \alpha_3) \cdots (x - \alpha_n)$$

where $c \in F_1^* \subseteq K^*$ and $\alpha_2, \ldots, \alpha_n \in K$.

Therefore, in $K[x]$:

$$f(x) = (x - \alpha)(x - \alpha_2) \cdots (x - \alpha_n)$$

splits completely. Since $K = F_1(\alpha_2, \ldots, \alpha_n) = F(\alpha, \alpha_2, \ldots, \alpha_n)$, the field K is generated over F by all roots of $f(x)$. Thus K is a splitting field for $f(x)$ over F.

For the degree bound:

$$[K : F] = [K : F_1] \cdot [F_1 : F] \leq (n-1)! \cdot n = n!.$$

This completes the induction. □

Remark 14.2.2. The bound $[K : F] \leq n!$ is often far from sharp. For example, for $f(x) = x^3 - 2$ over $\mathbb{Q}$, we have $n = 3$, so the bound gives $[K : \mathbb{Q}] \leq 6$. In fact, equality holds: $[K : \mathbb{Q}] = 6$. However, for $f(x) = x^3 - 1$, the splitting field has $[K : \mathbb{Q}] = 2 < 6$.

Example

Example 14.2.3 (Explicit Construction: $x^3 - 2$ over $\mathbb{F}_5$). Let's construct the splitting field of $f(x) = x^3 - 2$ over $\mathbb{F}_5$.

First, we check if $f(x)$ has roots in $\mathbb{F}_5$:

$$\begin{aligned} f(0) &= -2 \equiv 3 \pmod 5 \\ f(1) &= -1 \equiv 4 \pmod 5 \\ f(2) &= 6 \equiv 1 \pmod 5 \\ f(3) &= 25 \equiv 0 \pmod 5 \\ f(4) &= 62 \equiv 2 \pmod 5 \end{aligned}$$

So $\alpha_1 = 3$ is a root. We can factor:

$$x^3 - 2 = (x - 3)(x^2 + 3x + 4) \text{ in } \mathbb{F}_5[x].$$

Check if $g(x) = x^2 + 3x + 4$ has roots in $\mathbb{F}_5$:

$$\begin{aligned} g(0) &= 4 \\ g(1) &= 8 \equiv 3 \\ g(2) &= 14 \equiv 4 \\ g(3) &= 22 \equiv 2 \\ g(4) &= 32 \equiv 2 \end{aligned}$$

No roots in $\mathbb{F}_5$, so $g(x)$ is irreducible over $\mathbb{F}_5$.

Form $F_1 = \mathbb{F}_5[x]/(x^2 + 3x + 4)$. This is a field with $5^2 = 25$ elements. Let β be a root of $g(x)$ in F_1. Then the other root is $-3 - \beta$ (by Vieta's formulas). The splitting field is $K = \mathbb{F}_5(\beta) = F_1$ with $[K : \mathbb{F}_5] = 2$.

14.3 Uniqueness of Splitting Fields

Splitting fields are unique up to isomorphism. This fundamental result shows that the splitting field is a well-defined object.

14.3.1 Isomorphism Extension Theorem

The key to proving uniqueness is the following powerful theorem.

> **Theorem**
>
> **Theorem 14.3.1 (Isomorphism Extension).** Let $\varphi : F \to F'$ be an isomorphism of fields. Let $f(x) \in F[x]$ and let $f'(x) \in F'[x]$ be the image of $f(x)$ under φ (applying φ to each coefficient). Let K be a splitting field for $f(x)$ over F and K' a splitting field for $f'(x)$ over F'. Then φ extends to an isomorphism $\Phi : K \to K'$.

Proof. We prove this by induction on $[K : F]$.

Base case: $[K : F] = 1$. Then $K = F$ and $f(x)$ splits completely over F. This means all roots of $f(x)$ lie in F. Similarly, $K' = F'$ and all roots of $f'(x)$ lie in F'. The required extension is $\Phi = \varphi : K = F \to F' = K'$.

Inductive step: Assume $[K : F] > 1$. Since K is a splitting field, $f(x)$ does not split completely over F. Let $p(x)$ be an irreducible factor of $f(x)$ in $F[x]$ with $\deg(p) \geq 2$. Let $p'(x)$ be the image of $p(x)$ under φ; then $p'(x)$ is irreducible in $F'[x]$.

Let $\alpha \in K$ be a root of $p(x)$ and $\alpha' \in K'$ be a root of $p'(x)$. Such roots exist since K and K' are splitting fields. Consider the extensions $F(\alpha)$ and $F'(\alpha')$. We have:

- $F(\alpha) \cong F[x]/(p(x))$
- $F'(\alpha') \cong F'[x]/(p'(x))$

The isomorphism $\varphi : F \to F'$ extends to an isomorphism:

$$\bar{\varphi} : F(\alpha) \to F'(\alpha')$$

defined by $\bar{\varphi}(\alpha) = \alpha'$ and extending linearly:

$$\bar{\varphi}(a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1}) = \varphi(a_0) + \varphi(a_1)\alpha' + \cdots + \varphi(a_{n-1})(\alpha')^{n-1}$$

where $n = \deg(p)$.

Now, K is a splitting field for $f(x)$ over F, so it is also a splitting field for $f(x)$ over $F(\alpha)$. Similarly, K' is a splitting field for $f'(x)$ over $F'(\alpha')$. We have:

$$[K : F(\alpha)] = \frac{[K : F]}{[F(\alpha) : F]} < [K : F]$$

since $[F(\alpha) : F] = \deg(p) \geq 2$.

By the inductive hypothesis, $\bar{\varphi} : F(\alpha) \to F'(\alpha')$ extends to an isomorphism $\Phi : K \to K'$. This Φ is the required extension of φ. $\square$

> **Theorem**
>
> **Corollary 14.3.2 (Uniqueness of Splitting Fields).** Let F be a field and $f(x) \in F[x]$ a nonconstant polynomial. If K and K' are both splitting fields for $f(x)$ over F, then there exists an isomorphism $\Phi : K \to K'$ that is the identity on F.

Proof. Apply the Isomorphism Extension Theorem with $\varphi = \mathrm{id}_F : F \to F$, $f' = f$, and splitting fields K and K'. $\square$

> **Example**
>
> **Example 14.3.3 (Uniqueness for $x^3 - 2$).** The splitting field of $x^3 - 2$ over $\mathbb{Q}$ can be constructed as $\mathbb{Q}(\sqrt[3]{2}, \omega)$ where $\omega = e^{2\pi i/3}$. Alternatively, we could use $\omega' = e^{-2\pi i/3} = \omega^2$ and construct $\mathbb{Q}(\sqrt[3]{2}, \omega')$.
>
> The uniqueness theorem guarantees these are isomorphic. Indeed, the map:
>
> $$\Phi : \mathbb{Q}(\sqrt[3]{2}, \omega) \to \mathbb{Q}(\sqrt[3]{2}, \omega')$$
>
> defined by $\Phi(\sqrt[3]{2}) = \sqrt[3]{2}$ and $\Phi(\omega) = \omega'$ extends to an isomorphism.

14.3.2 Consequences of Uniqueness

Theorem

Corollary 14.3.4. If K is a splitting field for $f(x)$ over F, then any automorphism of K that fixes F permutes the roots of $f(x)$.

Proof. Let $\sigma \in \mathrm{Aut}(K)$ fix F. If $\alpha \in K$ is a root of $f(x)$, then:

$$0 = \sigma(0) = \sigma(f(\alpha)) = f(\sigma(\alpha))$$

where the last equality uses that σ fixes the coefficients of $f(x)$ (which lie in F). Thus $\sigma(\alpha)$ is also a root of $f(x)$. $\square$

This observation is fundamental to Galois theory, where we study the Galois group $\mathrm{Gal}(K/F)$ consisting of all automorphisms of K fixing F.

14.4 Normal Extensions

Splitting fields exemplify a broader class of extensions called normal extensions.

Definition

Definition 14.4.1 (Normal Extension). An algebraic extension K/F is *normal* if every irreducible polynomial $p(x) \in F[x]$ that has at least one root in K splits completely over K.

In other words, K/F is normal if: whenever $\alpha \in K$ is a root of an irreducible $p(x) \in F[x]$, then all roots of $p(x)$ lie in K.

Example

Example 14.4.2 (Normal Extension). $\mathbb{Q}(\sqrt{2},\sqrt{3})/\mathbb{Q}$ is normal.

To see this, let $p(x) \in \mathbb{Q}[x]$ be irreducible with a root $\alpha \in \mathbb{Q}(\sqrt{2},\sqrt{3})$. We need to show all roots of $p(x)$ lie in $\mathbb{Q}(\sqrt{2},\sqrt{3})$. Since $[\mathbb{Q}(\sqrt{2},\sqrt{3}) : \mathbb{Q}] = 4$ and α is a root of $p(x)$, we have $\deg(p) \mid 4$. The possible degrees are 1, 2, or 4.

- If $\deg(p) = 1$: $p(x) = x - \alpha$, already split.
- If $\deg(p) = 2$: The minimal polynomial of any element in $\mathbb{Q}(\sqrt{2},\sqrt{3})$ of degree 2 over $\mathbb{Q}$ has its roots related by conjugation, and both roots lie in $\mathbb{Q}(\sqrt{2},\sqrt{3})$.
- If $\deg(p) = 4$: Then $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{2},\sqrt{3})$, which is the splitting field of $(x^2-2)(x^2-3)$, so all conjugates of α lie in this field.

Example

Example 14.4.3 (Non-Normal Extension). $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is not normal.

The minimal polynomial of $\sqrt[3]{2}$ is $p(x) = x^3 - 2$, which has roots:

$$\sqrt[3]{2},\ \sqrt[3]{2}\omega,\ \sqrt[3]{2}\omega^2$$

where $\omega = e^{2\pi i/3}$. The complex roots $\sqrt[3]{2}\omega$ and $\sqrt[3]{2}\omega^2$ do not lie in $\mathbb{Q}(\sqrt[3]{2}) \subset \mathbb{R}$.

Theorem

Theorem 14.4.4 (Characterization of Normal Extensions). Let K/F be a finite extension. The following are equivalent:

1. K/F is normal.
2. K is the splitting field of some polynomial $f(x) \in F[x]$.
3. K is the splitting field of a set of polynomials in $F[x]$.
4. Every irreducible polynomial in $F[x]$ either has no roots in K or splits completely in K.
5. Every F-embedding $\sigma : K \to \overline{F}$ (into an algebraic closure) satisfies $\sigma(K) = K$.

Proof Sketch.

(ii) $\Rightarrow$ (i): Let K be the splitting field of $f(x) \in F[x]$. Let $p(x) \in F[x]$ be irreducible with a root $\alpha \in K$. Let β be any root of $p(x)$ in $\overline{F}$. By the Isomorphism Extension Theorem, there exists an isomorphism $\sigma : F(\alpha) \to F(\beta)$ fixing F and sending $\alpha \mapsto \beta$. This extends to an automorphism of the splitting field of $f(x)$ over $F(\alpha)$. Since splitting fields are unique up to isomorphism, σ extends to an automorphism of K, so $\beta = \sigma(\alpha) \in K$.

(i) $\Rightarrow$ (iii): Let $K = F(\alpha_1, \ldots, \alpha_n)$ (finite extension). Let $p_i(x)$ be the minimal polynomial of α_i over F. Since K/F is normal, each $p_i(x)$ splits completely in K. Let $f(x) = p_1(x) \cdots p_n(x)$. Then K is the splitting field of $\{p_1, \ldots, p_n\}$, or equivalently, of $f(x)$.

(iii) $\Rightarrow$ (ii): If K is the splitting field of $\{f_1, \ldots, f_m\}$, then K is the splitting field of $f(x) = f_1(x) \cdots f_m(x)$.

The equivalence with (iv) and (v) follows from the characterization of normal extensions in terms of embeddings. $\square$

Definition

Definition 14.4.5 (Normal Closure). Given an extension K/F, the *normal closure* of K over F is the smallest normal extension N of F containing K.

Example

Example 14.4.6 (Normal Closure of $\mathbb{Q}(\sqrt[3]{2})$). The normal closure of $\mathbb{Q}(\sqrt[3]{2})$ over $\mathbb{Q}$ is $\mathbb{Q}(\sqrt[3]{2}, \omega)$, the splitting field of $x^3 - 2$ over $\mathbb{Q}$. This is the smallest field containing $\mathbb{Q}(\sqrt[3]{2})$ that is normal over $\mathbb{Q}$.

We have:

$$\mathbb{Q} \subset \mathbb{Q}(\sqrt[3]{2}) \subset \mathbb{Q}(\sqrt[3]{2}, \omega)$$

with degrees $1 < 3 < 6$. The middle extension is not normal, but the full extension is normal.

Theorem

Proposition 14.4.7. The normal closure of K over F exists and is unique up to isomorphism. If $K = F(\alpha)$, then the normal closure is the splitting field of the minimal polynomial of α over F.

14.5 Degree of Splitting Fields

Computing the degree of a splitting field is often challenging but follows systematic principles.

14.5.1 Degree Bounds and Exact Computation

Theorem

Proposition 14.5.1. Let K be the splitting field of $f(x) \in F[x]$ where $\deg(f) = n$. Then:

$$[K : F] \mid n!.$$

Proof. From the existence theorem, we know $[K : F] \leq n!$. To show $[K : F] \mid n!$, we use the fact that $[K : F]$ equals the order of the Galois group $|\mathrm{Gal}(K/F)|$ when K/F is Galois (which holds for separable polynomials, the typical case). The Galois group embeds into the symmetric group S_n, which has order $n!$, so $|\mathrm{Gal}(K/F)| \mid n!$. The general case without separability assumptions requires more care, but the divisibility still holds. $\square$

Example

Example 14.5.2 (Degree Calculation: $\mathbb{Q}(\sqrt{2}, \sqrt[3]{2})$). Let $K = \mathbb{Q}(\sqrt{2}, \sqrt[3]{2})$. This is not the splitting field of any single irreducible polynomial, but we can still compute $[K : \mathbb{Q}]$.

Consider the tower:

$$\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt{2}, \sqrt[3]{2}).$$

We have $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$. For the second step, we need the degree of $\sqrt[3]{2}$ over $\mathbb{Q}(\sqrt{2})$.

The minimal polynomial of $\sqrt[3]{2}$ over $\mathbb{Q}$ is $m(x) = x^3 - 2$, which has degree 3. Does $m(x)$ remain irreducible over $\mathbb{Q}(\sqrt{2})$?

Suppose $m(x)$ factors over $\mathbb{Q}(\sqrt{2})$. Since $\deg(m) = 3$, it would have a linear factor, i.e., a root in $\mathbb{Q}(\sqrt{2})$. This means:

$$\sqrt[3]{2} = a + b\sqrt{2} \text{ for some } a, b \in \mathbb{Q}.$$

Cubing both sides:

$$2 = a^3 + 3a^2b\sqrt{2} + 6ab^2 + 2b^3\sqrt{2} = (a^3 + 6ab^2) + (3a^2b + 2b^3)\sqrt{2}.$$

Since $\{1, \sqrt{2}\}$ is a basis for $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$, we need:

$$\begin{aligned} a^3 + 6ab^2 &= 2 \\ 3a^2b + 2b^3 &= 0 \end{aligned}$$

From the second equation: $b(3a^2 + 2b^2) = 0$.

- If $b = 0$: From the first equation, $a^3 = 2$, so $a = \sqrt[3]{2} \notin \mathbb{Q}$. Contradiction.
- If $3a^2 + 2b^2 = 0$: Since $a, b \in \mathbb{Q}$ and squares are non-negative, this implies $a = b = 0$, contradicting $a^3 + 6ab^2 = 2$.

Therefore, $x^3 - 2$ is irreducible over $\mathbb{Q}(\sqrt{2})$, giving:

$$[\mathbb{Q}(\sqrt{2}, \sqrt[3]{2}) : \mathbb{Q}(\sqrt{2})] = 3.$$

Thus:

$$[K : \mathbb{Q}] = [\mathbb{Q}(\sqrt{2}, \sqrt[3]{2}) : \mathbb{Q}(\sqrt{2})] \cdot [\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 3 \cdot 2 = 6.$$

A basis for K over $\mathbb{Q}$ is:

$$\{1, \sqrt{2}, \sqrt[3]{2}, \sqrt[3]{4}, \sqrt{2}\sqrt[3]{2}, \sqrt{2}\sqrt[3]{4}\}.$$

Example

Example 14.5.3 (Degree Calculation: Splitting Field of $x^4 - 2$). Let K be the splitting field of $f(x) = x^4 - 2$ over $\mathbb{Q}$. The roots are:

$$\sqrt[4]{2}, i\sqrt[4]{2}, -\sqrt[4]{2}, -i\sqrt[4]{2}.$$

Let $\alpha = \sqrt[4]{2}$ (the positive real root). Then all roots are:

$$\alpha, i\alpha, -\alpha, -i\alpha = \alpha \cdot \{1, i, -1, -i\}.$$

Thus $K = \mathbb{Q}(\alpha, i)$.

Build the tower: $\mathbb{Q} \subseteq \mathbb{Q}(\alpha) \subseteq \mathbb{Q}(\alpha, i)$. The minimal polynomial of α over $\mathbb{Q}$ is $x^4 - 2$ (irreducible by Eisenstein with $p = 2$), so:

$$[\mathbb{Q}(\alpha) : \mathbb{Q}] = 4.$$

Is $i \in \mathbb{Q}(\alpha)$? Since $\mathbb{Q}(\alpha) \subset \mathbb{R}$ (as $\alpha \in \mathbb{R}$) and $i \notin \mathbb{R}$, we have $i \notin \mathbb{Q}(\alpha)$. The minimal polynomial of i over $\mathbb{Q}$ is $x^2 + 1$. Does it remain irreducible over $\mathbb{Q}(\alpha)$? If $x^2 + 1$ factors over $\mathbb{Q}(\alpha)$, it has a root in $\mathbb{Q}(\alpha)$, meaning $i \in \mathbb{Q}(\alpha)$, which we've shown is false. Therefore:

$$[\mathbb{Q}(\alpha, i) : \mathbb{Q}(\alpha)] = 2.$$

Thus:

$$[K : \mathbb{Q}] = 2 \cdot 4 = 8.$$

14.6 Composite Extensions

When we have two extensions $F(\alpha)$ and $F(\beta)$, understanding their composite $F(\alpha, \beta)$ is crucial.

Definition

Definition 14.6.1 (Composite Field). Given fields $F \subseteq K_1, K_2 \subseteq L$, the *composite* K_1K_2 is the smallest subfield of L containing both K_1 and K_2. If $K_1 = F(\alpha)$ and $K_2 = F(\beta)$, then $K_1K_2 = F(\alpha, \beta)$.

Theorem

Theorem 14.6.2 (Degree of Composite). Let K_1/F and K_2/F be finite extensions contained in some larger field L. Then:

$$[K_1K_2 : F] \mid [K_1 : F] \cdot [K_2 : F].$$

If $\gcd([K_1 : F], [K_2 : F]) = 1$, then:

$$[K_1K_2 : F] = [K_1 : F] \cdot [K_2 : F].$$

Proof. Consider the tower $F \subseteq K_1 \subseteq K_1K_2$. Then:

$$[K_1K_2 : F] = [K_1K_2 : K_1] \cdot [K_1 : F].$$

Similarly, $F \subseteq K_2 \subseteq K_1K_2$ gives:

$$[K_1K_2 : F] = [K_1K_2 : K_2] \cdot [K_2 : F].$$

From the first equality: $[K_1K_2 : K_1] = \frac{[K_1K_2:F]}{[K_1:F]}$. Now, K_2 embeds into K_1K_2 via the inclusion map. The degree $[K_1K_2 : K_1]$ is at most $[K_2 : F]$ (since K_2 is generated over F by finitely many elements, and adjoining these to K_1 gives K_1K_2). Therefore:

$$[K_1K_2 : F] = [K_1K_2 : K_1] \cdot [K_1 : F] \leq [K_2 : F] \cdot [K_1 : F].$$

For the coprime case: If $\gcd([K_1 : F], [K_2 : F]) = 1$, then from:

$$[K_1 : F] \mid [K_1K_2 : F]$$
$$[K_2 : F] \mid [K_1K_2 : F]$$

we get $\operatorname{lcm}([K_1 : F], [K_2 : F]) \mid [K_1K_2 : F]$. Since the gcd is 1, the lcm equals the product, so:

$$[K_1 : F] \cdot [K_2 : F] \mid [K_1 K_2 : F].$$

Combined with the upper bound, we get equality. □

Example

Example 14.6.3 (Composite: $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$). Let $K_1 = \mathbb{Q}(\sqrt{2})$ and $K_2 = \mathbb{Q}(\sqrt{3})$. Then:

$$[K_1 : \mathbb{Q}] = [K_2 : \mathbb{Q}] = 2.$$

Since $\gcd(2, 2) = 2 \neq 1$, we cannot immediately conclude the degrees multiply.

However, we showed earlier that $\sqrt{3} \notin \mathbb{Q}(\sqrt{2})$, so:

$$[K_1 K_2 : K_1] = [\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{2})] = 2.$$

Therefore:

$$[K_1 K_2 : \mathbb{Q}] = 2 \cdot 2 = 4 = [K_1 : \mathbb{Q}] \cdot [K_2 : \mathbb{Q}].$$

In this case, equality holds despite the gcd being 2, because $K_1 \cap K_2 = \mathbb{Q}$ (the fields are linearly disjoint).

Definition

Definition 14.6.4 (Linearly Disjoint Extensions). Extensions K_1/F and K_2/F (contained in some L) are *linearly disjoint* if $K_1 \cap K_2 = F$ and every F-linearly independent set in K_1 remains K_2-linearly independent in $K_1 K_2$.

Theorem

Proposition 14.6.5. If K_1 and K_2 are linearly disjoint over F, then:

$$[K_1 K_2 : F] = [K_1 : F] \cdot [K_2 : F].$$

Example

Example 14.6.6 (Non-Example: $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{8})$). Let $K_1 = \mathbb{Q}(\sqrt{2})$ and $K_2 = \mathbb{Q}(\sqrt{8}) = \mathbb{Q}(2\sqrt{2})$. Note that $\sqrt{8} = 2\sqrt{2} \in \mathbb{Q}(\sqrt{2})$, so:

$$K_2 \subseteq K_1.$$

Thus $K_1 K_2 = K_1$, giving:

$$[K_1 K_2 : \mathbb{Q}] = [K_1 : \mathbb{Q}] = 2 < 4 = [K_1 : \mathbb{Q}] \cdot [K_2 : \mathbb{Q}].$$

The extensions are not linearly disjoint since $K_1 \cap K_2 = K_2 \neq \mathbb{Q}$.

14.6.1 Lattice of Intermediate Fields

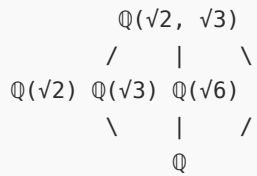
For a splitting field K/F, the intermediate fields $F \subseteq L \subseteq K$ form a lattice under inclusion. Understanding this lattice is central to Galois theory.

Example

Example 14.6.7 (Lattice for $\mathbb{Q}(\sqrt{2}, \sqrt{3})$). The extension $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$ has degree 4. The intermediate fields are:

- $\mathbb{Q}$ (degree 1)
- $\mathbb{Q}(\sqrt{2})$ (degree 2)
- $\mathbb{Q}(\sqrt{3})$ (degree 2)
- $\mathbb{Q}(\sqrt{6})$ (degree 2)
- $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ (degree 4)

The lattice structure:



Each path from $\mathbb{Q}$ to $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ has total degree $4 = 2 \cdot 2$.

Greek Meditation: Splitting and Unity

Heraclitus' Flux and Parmenides' Being Reconciled in the Splitting Field

The ancient Greeks debated the nature of change and permanence. Heraclitus declared "Everything flows," emphasizing ceaseless transformation. Parmenides countered that Being is eternal and indivisible, while multiplicity is illusion.

The splitting field synthesizes these opposing visions. A polynomial over $\mathbb{Q}$ represents a unified mathematical Being—a single irreducible essence. Yet to understand this Being fully, we must witness its splitting into roots, its fragmentation into linear factors.

In $\mathbb{Q}(\sqrt[3]{2}, \omega)$, the polynomial $x^3 - 2$ maintains its identity while revealing its triple nature. The field extension does not destroy the polynomial's unity but illuminates its hidden structure. Like Heraclitus' river that remains itself while constantly flowing, the polynomial is both one and many.

The uniqueness theorem echoes Parmenides: though we may construct the splitting field in various ways, its essential structure is invariant, eternal, beyond the accidents of construction. Yet the existence theorem acknowledges Heraclitus: to reach this eternal structure, we must undergo a process, adjoining roots step by step, flowing from F to K through intermediate fields.

The splitting field is where flux and permanence achieve harmony, where the Many and the One coexist without contradiction. It is the mathematical manifestation of Plato's Forms meeting Aristotle's actualizations—the perfect field where potential roots become actual, where the polynomial realizes its complete nature.

Zen Meditation: The Field Where All Roots Dwell

Completeness Without Grasping, Wholeness Without Attachment

A splitting field is complete: every root dwells within it. Yet this completeness is not achieved by grasping or accumulating, but by creating the precise conditions for wholeness to manifest.

The field does not chase after roots in the algebraic closure, anxiously gathering them. Rather, it expands just enough—no more, no less—to allow the polynomial's nature to fully unfold. This is *mushin* (無心): effortless action, purpose without strain.

Consider $\mathbb{Q}(\sqrt{2})$: to split $x^2 - 2$, we need only adjoin $\sqrt{2}$. We don't also adjoin π or e or infinitely many other elements. The field expands minimally yet completely. This is the Middle Way—neither ascetic restriction nor indulgent excess.

The uniqueness of the splitting field embodies *suchness* (tathatā): things as they fundamentally are, beyond the accidents of description. Whether we construct the splitting field starting with this root or that, by this path or another, we arrive at the same essential reality. The finger pointing at the moon may vary, but the moon remains.

In meditation, we seek not to add thoughts but to let the mind's true nature reveal itself. In algebra, we seek not to force a field into existence but to recognize the natural completion that the polynomial itself demands. When we stop forcing, when we allow the field to be exactly what it must be, splitting arises spontaneously.

The empty field, expanded by necessity, becomes full. Fullness that knows itself as emptiness: this is the splitting field, this is enlightenment.

14.7 Practice Problems

1. **Problem 14.1.** Find the splitting field K of $x^4 + 1$ over $\mathbb{Q}$ and determine $[K : \mathbb{Q}]$.

2. **Problem 14.2.** Prove that if K is the splitting field of a separable polynomial $f(x) \in F[x]$, then K/F is a Galois extension.
3. **Problem 14.3.** Let p be prime. Show that the splitting field of $x^p - x$ over $\mathbb{F}_p$ is $\mathbb{F}_p$ itself.
4. **Problem 14.4.** Find the splitting field of $x^4 - 5x^2 + 6$ over $\mathbb{Q}$ and determine all intermediate fields.
5. **Problem 14.5.** Show that $\mathbb{Q}(\sqrt{2} + \sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.
6. **Problem 14.6.** Let K be the splitting field of $x^n - 1$ over $\mathbb{Q}$. Determine K and $[K : \mathbb{Q}]$ for $n = 3, 4, 5, 6$.
7. **Problem 14.7.** Prove that if K/F is normal and L is an intermediate field, then K/L is normal, but L/F need not be normal. Give an example.
8. **Problem 14.8.** Find the normal closure of $\mathbb{Q}(\sqrt[4]{2})$ over $\mathbb{Q}$.
9. **Problem 14.9.** Let K be the splitting field of $x^6 - 4$ over $\mathbb{Q}$. Determine $[K : \mathbb{Q}]$ and describe the lattice of intermediate fields.
10. **Problem 14.10.** Show that if $[K_1 : F]$ and $[K_2 : F]$ are coprime, then $[K_1 K_2 : F] = [K_1 : F] \cdot [K_2 : F]$.
11. **Problem 14.11.** Determine whether $\mathbb{Q}(\sqrt[3]{2}, \sqrt{2})$ is a normal extension of $\mathbb{Q}$.
12. **Problem 14.12.** Find the splitting field of $x^3 - 3$ over $\mathbb{F}_5$ and determine its degree.
13. **Problem 14.13.** Let $\omega = e^{2\pi i/n}$ be a primitive n-th root of unity. Show that $\mathbb{Q}(\omega)$ is the splitting field of $x^n - 1$ over $\mathbb{Q}$.
14. **Problem 14.14.** Prove that the degree of the splitting field of $x^n - a$ over $\mathbb{Q}$ divides $n \cdot \varphi(n)$ where φ is Euler's totient function.
15. **Problem 14.15.** Show that if K and L are splitting fields over F (of possibly different polynomials), then KL is also a splitting field over F.
16. **Problem 14.16.** Let $p(x)$ be irreducible of degree n over F, and let K be the splitting field of $p(x)$. Show that $n \mid [K : F] \mid n!$.
17. **Problem 14.17.** Find the splitting field of $(x^2 - 2)(x^2 - 3)(x^2 - 5)$ over $\mathbb{Q}$ and determine its degree.
18. **Problem 14.18.** Prove that every finite extension of a finite field is normal.
19. **Problem 14.19.** Let $f(x) \in F[x]$ have degree n, and let K be its splitting field. Show that if $f(x)$ is separable, then $|\mathrm{Gal}(K/F)| = [K : F]$.

20. **Problem 14.20.** Determine all splitting fields of polynomials of degree 2 over $\mathbb{F}_3$.

14.8 Solutions to Practice Problems

Solution.

Solution 14.1

Problem: Find the splitting field K of x^4+1 over $\mathbb{Q}$ and determine $[K:\mathbb{Q}]$.

Solution:

Approach: We find the roots as 8th roots of unity, then determine the minimal field containing them.

Step 1: Find the roots of $x^4+1=0$.

We need $x^4=-1=e^{i\pi}$, so:

$$x = e^{i\pi(2k+1)/4} \text{ for } k=0,1,2,3$$

The roots are:

$$e^{i\pi/4}, \quad e^{3i\pi/4}, \quad e^{5i\pi/4}, \quad e^{7i\pi/4}$$

Let $\zeta = e^{i\pi/4} = \frac{1+i}{\sqrt{2}}$. Then the roots are $\zeta, \zeta^3, \zeta^5, \zeta^7$ where $\zeta^8=1$.

Step 2: Since all roots are powers of ζ, the splitting field is $K=\mathbb{Q}(\zeta)$.

Step 3: The polynomial $x^4+1=\Phi_8(x)$ (the 8th cyclotomic polynomial) is irreducible over $\mathbb{Q}$ with degree $\varphi(8)=4$.

Answer: $K=\mathbb{Q}(e^{i\pi/4})$ with $[K:\mathbb{Q}]=4$.

Solution.

Solution 14.2

Problem: Prove that if K is the splitting field of a separable polynomial $f(x)\in F[x]$, then K/F is a Galois extension.

Solution:

> ***Proof.***
>
> An extension is Galois if and only if it is both normal and separable.
>
> **Normal:** By definition, K is the splitting field of $f(x) \in F[x]$, so K/F is normal.
>
> **Separable:** Since $K = F(\alpha_1, \ldots, \alpha_n)$ where α_i are roots of the separable polynomial $f(x)$, each α_i is separable over F. The tower
>
> $$F \subseteq F(\alpha_1) \subseteq F(\alpha_1, \alpha_2) \subseteq \cdots \subseteq K$$
>
> consists of separable extensions, so K/F is separable. $\square$

∎

Solution.

Solution 14.3

Problem: Let p be prime. Show that the splitting field of $x^p - x$ over $\mathbb{F}_p$ is $\mathbb{F}_p$ itself.

Solution:

> ***Proof.***
>
> By Fermat's Little Theorem, for all $a \in \mathbb{F}_p$: $a^p \equiv a \pmod{p}$.
>
> Therefore, every element of $\mathbb{F}_p$ is a root of $x^p - x$. Since this polynomial has degree p and $|\mathbb{F}_p| = p$, all roots lie in $\mathbb{F}_p$:
>
> $$x^p - x = \prod_{a \in \mathbb{F}_p} (x - a)$$
>
> The splitting field is $\mathbb{F}_p$ itself. $\square$

∎

Solution.

Solution 14.4

Problem: Find the splitting field of $x^4 - 5x^2 + 6$ over $\mathbb{Q}$ and determine all intermediate fields.

Solution:

Step 1: Factor: $x^4 - 5x^2 + 6 = (x^2 - 2)(x^2 - 3)$.

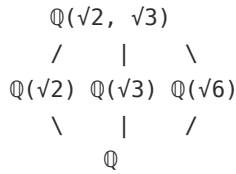
Step 2: The roots are $\pm\sqrt{2}, \pm\sqrt{3}$.

Step 3: The splitting field is $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ with $[K : \mathbb{Q}] = 4$.

Step 4: Intermediate fields of degree 2:

- $\mathbb{Q}(\sqrt{2})$
- $\mathbb{Q}(\sqrt{3})$
- $\mathbb{Q}(\sqrt{6})$

Answer: $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$, degree 4, with lattice:



Solution.

Solution 14.5

Problem: Show that $\mathbb{Q}(\sqrt{2} + \sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

Solution:

Proof.

Let $\alpha = \sqrt{2} + \sqrt{3}$.

(⊆): Clearly $\mathbb{Q}(\alpha) \subseteq \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

(⊇): We have $\alpha^2 = 5 + 2\sqrt{6}$, so $\sqrt{6} = \frac{\alpha^2 - 5}{2} \in \mathbb{Q}(\alpha)$.

Also, $\alpha \cdot \sqrt{6} = 3\sqrt{2} + 2\sqrt{3}$.

From $\alpha = \sqrt{2} + \sqrt{3}$ and $\alpha\sqrt{6} = 3\sqrt{2} + 2\sqrt{3}$, solving:

$$\sqrt{2} = \alpha(\sqrt{6} - 2) \in \mathbb{Q}(\alpha)$$

$$\sqrt{3} = \alpha - \sqrt{2} \in \mathbb{Q}(\alpha)$$

Therefore $\mathbb{Q}(\sqrt{2}, \sqrt{3}) \subseteq \mathbb{Q}(\alpha)$. $\square$

■

Solution.

Solution 14.6

Problem 14.6: Cyclotomic fields for n=3,4,5,6

Answer: Using $\varphi(n)$ for degrees:

- n=3: $K = \mathbb{Q}(e^{2\pi i/3})$, $[K : \mathbb{Q}] = 2$
- n=4: $K = \mathbb{Q}(i)$, $[K : \mathbb{Q}] = 2$
- n=5: $K = \mathbb{Q}(e^{2\pi i/5})$, $[K : \mathbb{Q}] = 4$
- n=6: $K = \mathbb{Q}(e^{\pi i/3})$, $[K : \mathbb{Q}] = 2$

Solution.

Solution 14.7

Problem 14.7: Normal extensions in towers

Proof.

Proof K/L normal: If K is the splitting field of $f(x) \in F[x]$, then K is also the splitting field of $f(x) \in L[x]$ for any intermediate L, so K/L is normal.

Example L/F not normal: Take $F = \mathbb{Q}$, $L = \mathbb{Q}(\sqrt[3]{2})$, $K = \mathbb{Q}(\sqrt[3]{2}, \omega)$ where $\omega = e^{2\pi i/3}$. Then $K/\mathbb{Q}$ is normal (splitting field of $x^3 - 2$), but $L/\mathbb{Q}$ is not (missing complex roots). $\square$

■

Solution.

Solution 14.8

Problem 14.8: Normal closure of $\mathbb{Q}(\sqrt[4]{2})$

Solution: The minimal polynomial is $x^4 - 2$ with roots $\pm\sqrt[4]{2}, \pm i\sqrt[4]{2}$.

Normal closure = splitting field = $\mathbb{Q}(\sqrt[4]{2}, i)$ with degree 8 over $\mathbb{Q}$.

Solution.

Solution 14.9

Problem 14.9: Splitting field of x^6-4

Solution: $x^6 - 4 = 0 \Rightarrow x^6 = 4 = 2^2$, so $x = \sqrt[3]{2} \cdot \zeta_6^k$ for $k = 0, \ldots, 5$.

Splitting field: $K = \mathbb{Q}(\sqrt[3]{2}, \omega)$ where $\omega = e^{2\pi i/3}$, with $[K : \mathbb{Q}] = 6$.

Lattice: $\mathbb{Q} \subset \mathbb{Q}(\omega), \mathbb{Q}(\sqrt[3]{2}) \subset \mathbb{Q}(\sqrt[3]{2}, \omega)$ with degrees 1, 2, 3, 6.

Solution.

Solution 14.10

Problem 14.10: Coprime degrees multiply

> ***Proof.***
>
> If $\gcd([K_1 : F], [K_2 : F]) = 1$, then both degrees divide $[K_1K_2 : F]$, so their lcm (which equals their product) divides $[K_1K_2 : F]$. But $[K_1K_2 : F] \leq [K_1 : F] \cdot [K_2 : F]$ always, so equality holds. $\square$

∎

Solution.

Solution 14.11

Problem 14.11: Q(∛2, √2) normality

Solution: First, $\sqrt{2} \notin \mathbb{Q}(\sqrt[3]{2})$ (real field, wrong degree), so $K = \mathbb{Q}(\sqrt[3]{2}, \sqrt{2})$ has degree 6.

The minimal polynomial $x^3 - 2$ of $\sqrt[3]{2}$ has complex roots $\omega\sqrt[3]{2}$ which are not in $K \subset \mathbb{R}$.

Answer: NOT normal.

Solution.

Solution 14.12

Problem 14.12: x³-3 over $\mathbb{F}_5$

Solution: Testing: $f(2) = 8 - 3 = 5 \equiv 0 \pmod 5$, so $x = 2$ is a root.

Factor: $x^3 - 3 = (x - 2)(x^2 + 2x + 4)$ in $\mathbb{F}_5[x]$.

The quadratic has no roots in $\mathbb{F}_5$, so is irreducible.

Answer: Splitting field is $\mathbb{F}_{25}$ with degree 2.

Solution.

Solution 14.13

Problem 14.13: $\mathbb{Q}(\omega)$ is splitting field of xⁿ-1

> ***Proof.***
>
> If $\omega = e^{2\pi i/n}$ is primitive, then all n-th roots of unity are $1, \omega, \omega^2, \ldots, \omega^{n-1} \in \mathbb{Q}(\omega)$.
>
> Conversely, $\mathbb{Q}(\omega)$ is minimal containing all roots, so it's the splitting field. $\square$

∎

Solution.

Solution 14.14

Problem 14.14: Degree of splitting field of $x^n - a$ divides $n \cdot \varphi(n)$

> ***Proof.***
>
> Splitting field $K = \mathbb{Q}(\sqrt[n]{a}, \zeta_n)$ where ζ_n is a primitive n-th root of unity.
>
> We have $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \varphi(n)$ and $[K : \mathbb{Q}(\zeta_n)] \mid n$.
>
> Therefore $[K : \mathbb{Q}] \mid n \cdot \varphi(n)$. $\square$

∎

Solution.

Solution 14.15

Problem 14.15: Composite of splitting fields

> ***Proof.***
>
> If K splits $f(x)$ and L splits $g(x)$, then KL contains all roots of both, so $f(x)g(x)$ splits in KL.
>
> Also, KL is generated by roots of $f(x)g(x)$, so KL is the splitting field of $f(x)g(x)$. $\square$

∎

Solution.

Solution 14.16

Problem 14.16: n | [K:F] | n! for irreducible p(x) of degree n

> ***Proof.***
>
> **Left divisibility:** If α is a root, then $[F(\alpha) : F] = n$ and $F \subseteq F(\alpha) \subseteq K$, so $n \mid [K : F]$.
>
> **Right divisibility:** Splitting fields of degree-n polynomials have degree dividing $n!$ by the existence theorem. □

∎

Solution.

Solution 14.17

Problem 14.17: Splitting field of $(x^2-2)(x^2-3)(x^2-5)$

Solution: Roots are $\pm\sqrt{2}, \pm\sqrt{3}, \pm\sqrt{5}$.

Splitting field: $K = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$.

Using towers and checking $\sqrt{3} \notin \mathbb{Q}(\sqrt{2})$ and $\sqrt{5} \notin \mathbb{Q}(\sqrt{2}, \sqrt{3})$:

Answer: $[K : \mathbb{Q}] = 2 \cdot 2 \cdot 2 = 8$.

Solution.

Solution 14.18

Problem 14.18: All finite field extensions are normal

> ***Proof.***
>
> Every finite field extension $\mathbb{F}_{q^n}/\mathbb{F}_q$ is the splitting field of $x^{q^n} - x$ over $\mathbb{F}_q$ (since every element $\alpha \in \mathbb{F}_{q^n}$ satisfies $\alpha^{q^n} = \alpha$).
>
> Therefore, it's normal by definition. □

∎

Solution.

Solution 14.19

Problem 14.19: Separable splitting fields are Galois

> ***Proof.***
>
> If K is the splitting field of separable $f(x)$, then K/F is both normal (it's a splitting field) and separable (generated by roots of a separable polynomial).
>
> Therefore K/F is Galois, and by the Fundamental Theorem: $|\mathrm{Gal}(K/F)| = [K : F]$. $\square$

∎

> ***Solution.***

Solution 14.20

Problem 14.20: All splitting fields of degree-2 polynomials over $\mathbb{F}_3$

Solution: A degree-2 polynomial over $\mathbb{F}_3$ either:

- Splits in $\mathbb{F}_3$ (reducible) → splitting field is $\mathbb{F}_3$
- Is irreducible → splitting field is $\mathbb{F}_9$

Answer: Exactly two splitting fields: $\mathbb{F}_3$ and $\mathbb{F}_9$.

Chapter 15: Finite Fields

"In finite perfection lies the mirror of the infinite, where closure reveals cyclicity, and order reflects eternity."

Finite fields are among the most beautiful and practical structures in mathematics. Despite their finiteness, they possess rich algebraic properties and exhibit remarkable regularity. This chapter explores the classification, construction, and properties of finite fields, revealing their elegant structure and widespread applications.

15.1 Structure of Finite Fields

Finite fields have a particularly rigid structure: their orders must be prime powers, and for each prime power there is a unique finite field up to isomorphism.

15.1.1 Classification by Order

> **Theorem**
>
> **Theorem 15.1.1 (Characteristic of a Finite Field).** Every finite field has characteristic p for some prime p.

Proof. Let F be a finite field. The characteristic of F is either 0 or a prime p. If $\text{char}(F) = 0$, then F contains a subfield isomorphic to $\mathbb{Q}$. But $\mathbb{Q}$ is infinite, contradicting the finiteness of F. Therefore $\text{char}(F) = p$ for some prime p. $\square$

> **Theorem**
>
> **Theorem 15.1.2 (Order is a Prime Power).** If F is a finite field, then $|F| = p^n$ for some prime p and positive integer n.

Proof. Let $\text{char}(F) = p$. Then F contains a prime subfield isomorphic to $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, which has p elements. Thus F is an extension of $\mathbb{F}_p$. Since F is finite, the extension $F/\mathbb{F}_p$ is finite, say $[F : \mathbb{F}_p] = n$. As a vector space over $\mathbb{F}_p$, the field F has dimension n. A basis has n elements, and each element of F can be written uniquely as:

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$

where $v_1, \ldots, v_n$ is a basis and $a_i \in \mathbb{F}_p$. There are p choices for each a_i, giving p^n total elements in F. Therefore $|F| = p^n$. $\square$

> **Theorem**
>
> **Theorem 15.1.3 (Existence and Uniqueness).** For every prime p and positive integer n, there exists a unique (up to isomorphism) finite field with p^n elements, denoted $\mathbb{F}_{p^n}$.

We prove existence and uniqueness in Section 15.2.

15.1.2 Subfield Structure

> **Theorem**
>
> **Theorem 15.1.4 (Subfield Criterion).** Let $F = \mathbb{F}_{p^n}$. Then F has a subfield isomorphic to $\mathbb{F}_{p^m}$ if and only if $m \mid n$. Moreover, this subfield is unique.

Proof. ($\Leftarrow$) Suppose $m \mid n$, say $n = mk$. Consider:

$$K = \{x \in F : x^{p^m} = x\}.$$

We claim K is a subfield of F with $|K| = p^m$.

Subfield: If $x, y \in K$, then:

$$\begin{aligned}(x+y)^{p^m} &= x^{p^m} + y^{p^m} = x + y \quad \text{(Frobenius)} \\ (xy)^{p^m} &= x^{p^m} y^{p^m} = xy \\ (x^{-1})^{p^m} &= (x^{p^m})^{-1} = x^{-1} \quad (x \neq 0)\end{aligned}$$

So K is closed under addition, multiplication, and inverses. Thus K is a subfield.

Order: The elements of K are precisely the roots of $f(x) = x^{p^m} - x \in \mathbb{F}_p[x]$. The derivative is $f'(x) = p^m x^{p^m - 1} - 1 = -1 \neq 0$ (since $p^m = 0$ in $\mathbb{F}_p$), so $f(x)$ is separable. Therefore $f(x)$ has exactly p^m distinct roots in its splitting field. Since F contains all roots of $x^{p^n} - x$ and $x^{p^m} - x$ divides $x^{p^n} - x$ (because $p^m - 1 \mid p^n - 1$), all p^m roots lie in F. Thus $|K| = p^m$ and $K \cong \mathbb{F}_{p^m}$.

($\Rightarrow$) Suppose F has a subfield K with $|K| = p^m$. Then $K \cong \mathbb{F}_{p^m}$, so:

$$[F : K] \cdot [K : \mathbb{F}_p] = [F : \mathbb{F}_p]$$

$$m \cdot [F : K] = n$$

Therefore $m \mid n$.

Uniqueness: Any subfield of order p^m consists of the roots of $x^{p^m} - x$, which are uniquely determined. □

Example

Example 15.1.5 (Subfields of $\mathbb{F}_{2^{12}}$). The field $\mathbb{F}_{2^{12}}$ has subfields $\mathbb{F}_{2^d}$ for each divisor d of 12.

Divisors of 12: 1, 2, 3, 4, 6, 12. Subfields: $\mathbb{F}_2, \mathbb{F}_4, \mathbb{F}_8, \mathbb{F}_{16}, \mathbb{F}_{64}, \mathbb{F}_{4096}$. The subfield lattice:

```
          𝔽₄₀₉₆ = 𝔽₂₁₂
            [3]  [2]
          𝔽₁₆    𝔽₆₄
        [2] [2] [3] [2]
          𝔽₄      𝔽₈
           [2]   [3]
             𝔽₂
```

Note: $\mathbb{F}_4 \not\subseteq \mathbb{F}_8$ since $2 \nmid 3$, but both embed in $\mathbb{F}_{16}$ and higher powers.

15.2 Construction of Finite Fields

We now prove the existence of finite fields and provide explicit constructions.

Theorem

Theorem 15.2.1 (Construction via Splitting Fields). For any prime p and positive integer n, the splitting field of $f(x) = x^{p^n} - x$ over $\mathbb{F}_p$ is a field with exactly p^n elements.

Proof. Let K be the splitting field of $f(x) = x^{p^n} - x$ over $\mathbb{F}_p$. Let:

$$S = \{\alpha \in K : f(\alpha) = 0\} = \{\alpha \in K : \alpha^{p^n} = \alpha\}.$$

We claim S is a subfield of K with $|S| = p^n$.

S is a subfield:

- $0, 1 \in S$ since $0^{p^n} = 0$ and $1^{p^n} = 1$.
- If $\alpha, \beta \in S$: $(\alpha + \beta)^{p^n} = \alpha^{p^n} + \beta^{p^n} = \alpha + \beta$ (Frobenius).

- If $\alpha, \beta \in S$: $(\alpha\beta)^{p^n} = \alpha^{p^n}\beta^{p^n} = \alpha\beta$.
- If $\alpha \in S^*$: $(\alpha^{-1})^{p^n} = (\alpha^{p^n})^{-1} = \alpha^{-1}$.

Thus S is closed under field operations.

$|S| = p^n$: The polynomial $f(x) = x^{p^n} - x$ has degree p^n, so it has at most p^n roots in K. Moreover, $f'(x) = p^n x^{p^n-1} - 1 = -1$ in $\mathbb{F}_p[x]$ (since $p = 0$ in $\mathbb{F}_p$). Thus $\gcd(f, f') = 1$, so f has no repeated roots. Therefore f has exactly p^n distinct roots in K, giving $|S| = p^n$. Since K is the splitting field of $f(x)$, it is generated over $\mathbb{F}_p$ by the roots of $f(x)$, which all lie in S. Thus $K = S$, and K is a field with p^n elements. $\square$

15.2.1 Explicit Constructions

To construct $\mathbb{F}_{p^n}$ explicitly, we find an irreducible polynomial of degree n over $\mathbb{F}_p$ and form the quotient ring.

> **Theorem**
>
> **Theorem 15.2.2.** If $p(x) \in \mathbb{F}_p[x]$ is irreducible of degree n, then:
>
> $$\mathbb{F}_{p^n} \cong \mathbb{F}_p[x]/(p(x)).$$

Proof. The quotient $K = \mathbb{F}_p[x]/(p(x))$ is a field (since $p(x)$ is irreducible). As a vector space over $\mathbb{F}_p$, it has basis $\{1, \bar{x}, \bar{x}^2, \ldots, \bar{x}^{n-1}\}$ (where $\bar{x}$ is the image of x). Thus $\dim_{\mathbb{F}_p}(K) = n$, giving $|K| = p^n$. By uniqueness, $K \cong \mathbb{F}_{p^n}$. $\square$

Example

Example 15.2.3 (Construction of $\mathbb{F}_4$). We need an irreducible polynomial of degree 2 over $\mathbb{F}_2 = \{0, 1\}$. Consider $p(x) = x^2 + x + 1$:

$$p(0) = 0 + 0 + 1 = 1 \neq 0$$
$$p(1) = 1 + 1 + 1 = 1 \neq 0$$

No roots in $\mathbb{F}_2$, so $p(x)$ is irreducible.

Form $\mathbb{F}_4 = \mathbb{F}_2[x]/(x^2 + x + 1)$. Let α denote the image of x in the quotient. Then:

$$\alpha^2 + \alpha + 1 = 0 \Rightarrow \alpha^2 = \alpha + 1.$$

The elements of $\mathbb{F}_4$ are:

$$0 = 0$$
$$1 = 1$$
$$\alpha = \alpha$$
$$\alpha + 1 = \alpha^2 \text{ (by the relation)}$$

Addition and multiplication tables:

+	0	1	α	α+1
0	0	1	α	α+1
1	1	0	α+1	α
α	α	α+1	0	1
α+1	α+1	α	1	0

·	0	1	α	α+1
0	0	0	0	0
1	0	1	α	α+1
α	0	α	α+1	1
α+1	0	α+1	1	α

Note: $\alpha(\alpha + 1) = \alpha^2 + \alpha = (\alpha + 1) + \alpha = 1$ (using $\alpha^2 = \alpha + 1$).

Example

Example 15.2.4 (Construction of $\mathbb{F}_8$). We need an irreducible polynomial of degree 3 over $\mathbb{F}_2$. Consider $p(x) = x^3 + x + 1$:

$$\begin{aligned} p(0) &= 1 \\ p(1) &= 1 + 1 + 1 = 1 \end{aligned}$$

No roots, so irreducible (degree 3 polynomial over a field is irreducible iff it has no roots).

Form $\mathbb{F}_8 = \mathbb{F}_2[x]/(x^3 + x + 1)$. Let α be the image of x. Then $\alpha^3 = \alpha + 1$. Elements: $0, 1, \alpha, \alpha + 1, \alpha^2, \alpha^2 + 1, \alpha^2 + \alpha, \alpha^2 + \alpha + 1$ (8 elements). Powers of α:

$$\begin{aligned} \alpha^0 &= 1 \\ \alpha^1 &= \alpha \\ \alpha^2 &= \alpha^2 \\ \alpha^3 &= \alpha + 1 \\ \alpha^4 &= \alpha^2 + \alpha \\ \alpha^5 &= \alpha^3 + \alpha^2 = (\alpha + 1) + \alpha^2 = \alpha^2 + \alpha + 1 \\ \alpha^6 &= \alpha^3 + \alpha^2 + \alpha = (\alpha + 1) + \alpha^2 + \alpha = \alpha^2 + 1 \\ \alpha^7 &= \alpha^3 + \alpha = (\alpha + 1) + \alpha = 1 \end{aligned}$$

So α has order 7, and $\mathbb{F}_8^* = \langle \alpha \rangle$ is cyclic of order 7.

Example

Example 15.2.5 (Construction of $\mathbb{F}_9$). We need an irreducible polynomial of degree 2 over $\mathbb{F}_3 = \{0, 1, 2\}$. Consider $p(x) = x^2 + 1$:

$$\begin{aligned} p(0) &= 1 \\ p(1) &= 1 + 1 = 2 \\ p(2) &= 4 + 1 = 5 \equiv 2 \pmod{3} \end{aligned}$$

No roots, so irreducible.

Form $\mathbb{F}_9 = \mathbb{F}_3[x]/(x^2 + 1)$. Let α be the image of x. Then $\alpha^2 = -1 = 2$ (in $\mathbb{F}_3$). Elements: $0, 1, 2, \alpha, \alpha + 1, \alpha + 2, 2\alpha, 2\alpha + 1, 2\alpha + 2$ (9 elements). Note: α satisfies $\alpha^2 = 2$, behaving like i in characteristic 3.

Example

Example 15.2.6 (Construction of $\mathbb{F}_{27}$). We need an irreducible polynomial of degree 3 over $\mathbb{F}_3$. Consider $p(x) = x^3 + 2x + 1$. Check for roots:

$$\begin{aligned} p(0) &= 1 \\ p(1) &= 1 + 2 + 1 = 4 \equiv 1 \\ p(2) &= 8 + 4 + 1 = 13 \equiv 1 \end{aligned}$$

No roots, so irreducible (degree 3).

Form $\mathbb{F}_{27} = \mathbb{F}_3[x]/(x^3 + 2x + 1)$. Let α be the image of x. Then:

$$\alpha^3 = -2\alpha - 1 = \alpha + 2 \ (\text{in } \mathbb{F}_3, -2 = 1).$$

The field has $27 = 3^3$ elements, with basis $\{1, \alpha, \alpha^2\}$ over $\mathbb{F}_3$.

15.3 Multiplicative Group $\mathbb{F}_{p^n}^*$

The multiplicative group of a finite field has a beautiful structure: it is always cyclic.

Theorem

Theorem 15.3.1 (Multiplicative Group is Cyclic). For any prime power $q = p^n$, the multiplicative group $\mathbb{F}_q^*$ is cyclic of order $q - 1$.

Proof. Let $F = \mathbb{F}_q$ where $q = p^n$. We know $|F^*| = q - 1$. It suffices to show F^* is cyclic. We use the following lemma: A finite abelian group G is cyclic if and only if for every divisor d of $|G|$, there are at most d solutions to $x^d = 1$ in G. Since F^* is abelian, we need to show: for every divisor d of $q - 1$, the polynomial $x^d - 1$ has at most d roots in F. This is immediate from field theory: a polynomial of degree d over a field has at most d roots. Therefore F^* is cyclic. $\square$

(Alternative direct proof using structure theorem for finite abelian groups is also possible.)

Definition

Definition 15.3.2 (Primitive Element). An element $\alpha \in \mathbb{F}_q^*$ is called *primitive* if α generates $\mathbb{F}_q^*$, i.e., if $\operatorname{ord}(\alpha) = q - 1$.

Example

Example 15.3.3 (Primitive Elements in $\mathbb{F}_4$). We have $\mathbb{F}_4^* = \{1, \alpha, \alpha+1\}$ with $|\mathbb{F}_4^*| = 3$.

Check orders:

$\operatorname{ord}(1) = 1$
$\operatorname{ord}(\alpha) = ? \quad \alpha^2 = \alpha + 1, \quad \alpha^3 = \alpha \cdot \alpha^2 = \alpha(\alpha+1) = \alpha^2 + \alpha = (\alpha+1) + \alpha = 1$

So $\operatorname{ord}(\alpha) = 3$. Thus α is primitive.

Similarly, $\alpha + 1$ is also primitive since $(\alpha+1)^3 = 1$ but $(\alpha+1)^1 = \alpha + 1 \neq 1$. Actually, $\alpha + 1 = \alpha^2$, so $(\alpha+1)^2 = \alpha^4 = \alpha$, and $(\alpha+1)^3 = \alpha^6 = 1$. Both nonidentity elements are primitive (as expected: $\varphi(3) = 2$ primitive elements).

Theorem

Theorem 15.3.4 (Number of Primitive Elements). The number of primitive elements in $\mathbb{F}_q^*$ is $\varphi(q-1)$, where φ is Euler's totient function.

Proof. Since $\mathbb{F}_q^* \cong \mathbb{Z}/(q-1)\mathbb{Z}$ (cyclic of order $q-1$), the number of generators equals the number of generators of $\mathbb{Z}/(q-1)\mathbb{Z}$, which is $\varphi(q-1)$. □

15.3.1 Finding Primitive Elements

Example

Example 15.3.5 (Finding Primitives in $\mathbb{F}_{11}$). We have $\mathbb{F}_{11}^* = \{1, 2, 3, \ldots, 10\}$ with order 10. To check if α is primitive, we need $\operatorname{ord}(\alpha) = 10$. By Lagrange, $\operatorname{ord}(\alpha) \mid 10$, so possible orders are 1, 2, 5, 10.

An element has order 10 iff it doesn't have order 1, 2, or 5. Equivalently: $\alpha^2 \not\equiv 1$, $\alpha^5 \not\equiv 1 \pmod{11}$. Check $\alpha = 2$:

$$2^2 = 4$$
$$2^5 = 32 \equiv 10 \equiv -1 \pmod{11}$$

Both $\neq 1$, so $\operatorname{ord}(2) = 10$. Thus 2 is primitive.

The primitive elements are: 2, 6, 7, 8 (there are $\varphi(10) = 4$ of them).

15.4 Frobenius Automorphism

The Frobenius map is a fundamental automorphism of finite fields.

> **Definition**
>
> **Definition 15.4.1 (Frobenius Endomorphism).** Let $F = \mathbb{F}_{p^n}$. The *Frobenius endomorphism* is the map:
>
> $$\varphi : F \to F, \quad \varphi(x) = x^p.$$

> **Theorem**
>
> **Theorem 15.4.2 (Frobenius is an Automorphism).** The Frobenius map $\varphi(x) = x^p$ is a field automorphism of $\mathbb{F}_{p^n}$ that fixes $\mathbb{F}_p$ pointwise.

Proof. **Homomorphism:**

- $\varphi(1) = 1^p = 1$.
- $\varphi(xy) = (xy)^p = x^p y^p = \varphi(x)\varphi(y)$.
- $\varphi(x + y) = (x + y)^p = x^p + y^p$ (by Frobenius property in characteristic p: all binomial coefficients $\binom{p}{k}$ for $0 < k < p$ are divisible by p).

Injective: If $\varphi(x) = \varphi(y)$, then $x^p = y^p$, so $(x - y)^p = x^p - y^p = 0$. In a field, this implies $x - y = 0$, so $x = y$.

Surjective: Since F is finite and φ is injective, it is automatically surjective.

Fixes $\mathbb{F}_p$: For $a \in \mathbb{F}_p$, Fermat's Little Theorem gives $a^p = a$, so $\varphi(a) = a$. $\square$

> **Theorem**
>
> **Proposition 15.4.3 (Order of Frobenius).** Let φ be the Frobenius automorphism of $\mathbb{F}_{p^n}$. Then $\varphi^n = \mathrm{id}$, and n is the smallest positive integer with this property.

Proof. We have:

$$\varphi^n(x) = x^{p^n}.$$

For any $x \in \mathbb{F}_{p^n}$, we have $x^{p^n} = x$ (by definition of $\mathbb{F}_{p^n}$ as roots of $x^{p^n} - x$). Thus $\varphi^n = \mathrm{id}$. For minimality, suppose $\varphi^m = \mathrm{id}$ for some $0 < m < n$. Then $x^{p^m} = x$ for all

$x \in \mathbb{F}_{p^n}$. This means $\mathbb{F}_{p^n} \subseteq \mathbb{F}_{p^m}$, contradicting $|\mathbb{F}_{p^n}| = p^n > p^m = |\mathbb{F}_{p^m}|$. $\square$

> **Theorem**
>
> **Theorem 15.4.4 (Fixed Field of Frobenius).** The fixed field of φ (elements x with $\varphi(x) = x$) is precisely $\mathbb{F}_p$.

Proof. An element $x \in \mathbb{F}_{p^n}$ is fixed by φ iff $x^p = x$, i.e., x is a root of $x^p - x$. The polynomial $x^p - x$ has exactly p roots in $\mathbb{F}_{p^n}$, which are precisely the elements of $\mathbb{F}_p$. Therefore $\mathrm{Fix}(\varphi) = \mathbb{F}_p$. $\square$

> **Theorem**
>
> **Theorem 15.4.5 (Galois Group of Finite Fields).** Let $F = \mathbb{F}_{p^n}$. Then:
>
> $$\mathrm{Gal}(F/\mathbb{F}_p) = \langle \varphi \rangle \cong \mathbb{Z}/n\mathbb{Z}$$
>
> where φ is the Frobenius automorphism.

Proof. We know $\varphi^n = \mathrm{id}$ and $\varphi^k \neq \mathrm{id}$ for $0 < k < n$. Thus $\langle \varphi \rangle$ is cyclic of order n. By the fundamental theorem of Galois theory (and since $F/\mathbb{F}_p$ is Galois):

$$|\mathrm{Gal}(F/\mathbb{F}_p)| = [F : \mathbb{F}_p] = n.$$

Since $\varphi \in \mathrm{Gal}(F/\mathbb{F}_p)$ and $|\langle \varphi \rangle| = n = |\mathrm{Gal}(F/\mathbb{F}_p)|$, we have:

$$\mathrm{Gal}(F/\mathbb{F}_p) = \langle \varphi \rangle \cong \mathbb{Z}/n\mathbb{Z}. \quad \square$$

Example

Example 15.4.6 (Galois Group of $\mathbb{F}_8/\mathbb{F}_2$). We have $\mathbb{F}_8 = \mathbb{F}_{2^3}$, so $\mathrm{Gal}(\mathbb{F}_8/\mathbb{F}_2) \cong \mathbb{Z}/3\mathbb{Z}$ generated by $\varphi(x) = x^2$.

The automorphisms are:

- $\varphi^0 = \mathrm{id}$
- $\varphi^1 : x \mapsto x^2$
- $\varphi^2 : x \mapsto x^4$

Let α be a root of $x^3 + x + 1$ in $\mathbb{F}_8$. Then:

$$\begin{aligned} \varphi(\alpha) &= \alpha^2 \\ \varphi^2(\alpha) &= \alpha^4 \\ \varphi^3(\alpha) &= \alpha^8 = \alpha \end{aligned}$$

The three roots of $x^3 + x + 1$ are $\alpha, \alpha^2, \alpha^4$, and Frobenius cycles through them.

15.5 Irreducible Polynomials over Finite Fields

Understanding irreducible polynomials over finite fields is crucial for constructing field extensions and applications.

15.5.1 Counting Irreducible Polynomials

Theorem

Theorem 15.5.1 (Number of Irreducibles). Let $N_q(n)$ denote the number of monic irreducible polynomials of degree n over $\mathbb{F}_q$. Then:

$$N_q(n) = \frac{1}{n} \sum_{d \mid n} \mu(d) q^{n/d}$$

where μ is the Möbius function.

For small values:

$$N_q(1) = q$$
$$N_q(2) = \frac{q^2 - q}{2}$$
$$N_q(3) = \frac{q^3 - q}{3}$$
$$N_q(4) = \frac{q^4 - q^2}{4}$$

Example

Example 15.5.2 (Irreducibles over $\mathbb{F}_2$).

- Degree 1: $N_2(1) = 2$ polynomials: $x, x+1$.
- Degree 2: $N_2(2) = \frac{4-2}{2} = 1$ polynomial: $x^2 + x + 1$ (can verify: x^2, x^2+1 reducible).
- Degree 3: $N_2(3) = \frac{8-2}{3} = 2$ polynomials: $x^3 + x + 1, x^3 + x^2 + 1$.
- Degree 4: $N_2(4) = \frac{16-4}{4} = 3$ polynomials: $x^4 + x + 1, x^4 + x^3 + 1, x^4 + x^3 + x^2 + x + 1$.

Theorem

Theorem 15.5.3 (Existence of Irreducibles). For any prime power q and positive integer n, there exists an irreducible polynomial of degree n over $\mathbb{F}_q$. In fact, $N_q(n) \geq 1$ for all $n \geq 1$.

Proof. From the formula, for $n \geq 1$:

$$N_q(n) = \frac{1}{n}\sum_{d \mid n} \mu(d) q^{n/d} \geq \frac{1}{n}(q^n - \text{lower order terms}) > 0.$$

More precisely, the dominant term is $\frac{q^n}{n}$, which exceeds the sum of all other terms for $q, n \geq 2$. $\square$

15.5.2 Factorization Algorithms

Example

Example 15.5.4 (Berlekamp's Algorithm Overview). Berlekamp's algorithm factors polynomials over finite fields in polynomial time. The basic idea:

Given $f(x) \in \mathbb{F}_q[x]$, construct the Berlekamp matrix:

$$Q = [q_{ij}]$$

where q_{ij} is the coefficient of x^i in $x^{qj} \bmod f(x)$. The null space of $Q - I$ has dimension equal to the number of irreducible factors of $f(x)$. This algorithm is widely used in computer algebra systems.

15.6 Applications

Finite fields have numerous applications across mathematics and computer science.

15.6.1 Coding Theory

Reed-Solomon Codes: Used in CDs, DVDs, QR codes, and deep-space communication. A Reed-Solomon code over $\mathbb{F}_q$ with parameters $[n, k, n-k+1]$ can:

- Encode k symbols into n symbols.
- Correct up to $\lfloor (n-k)/2 \rfloor$ errors.

Example: CD audio uses RS(32,28) over $\mathbb{F}_{256}$, correcting 2 errors per block.

BCH Codes: Generalization of Reed-Solomon codes, using minimal polynomials over extension fields.

15.6.2 Cryptography

AES (Advanced Encryption Standard): Operates in $\mathbb{F}_{256} = \mathbb{F}_{2^8}$. The S-box uses the multiplicative inverse in $\mathbb{F}_{256}$ (constructed via $\mathbb{F}_2[x]/(x^8 + x^4 + x^3 + x + 1)$).

Elliptic Curve Cryptography: Elliptic curves over finite fields $\mathbb{F}_p$ or $\mathbb{F}_{2^n}$ provide:

- Shorter keys than RSA for equivalent security.
- Efficient computations.

Example: Bitcoin uses curve secp256k1 over $\mathbb{F}_p$ where $p = 2^{256} - 2^{32} - 977$.

Diffie-Hellman Key Exchange: Based on discrete logarithm in $\mathbb{F}_p^*$. Given g (generator) and g^a, g^b, computing g^{ab} is easy if you know a or b, but hard given only g^a and g^b (discrete log problem).

15.6.3 Other Applications

Combinatorics: Finite projective planes, Latin squares, block designs.

Pseudorandom Generation: Linear feedback shift registers (LFSRs) over $\mathbb{F}_2$ generate maximal-length sequences.

Computer Algebra: Polynomial GCD algorithms, modular arithmetic, fast Fourier transforms over finite fields.

> *Greek Meditation: Finite Perfection*
>
> **The Cosmos in Microcosm, The Infinite in the Finite**
>
> The ancient Greeks sought perfection in both the infinite and the finite. The Pythagoreans revered finite numbers yet believed in the infinity of the cosmos. Plato's Forms were eternal and infinite, yet each particular instantiation was finite and bounded.
>
> A finite field embodies this paradox. With only p^n elements, it is emphatically finite, countable, graspable. Yet within this finite structure blooms an infinity of relationships: polynomial equations, algebraic extensions, automorphisms, symmetries.
>
> The cyclicity of $\mathbb{F}_q^*$ reflects the Greek conception of cosmic cycles—the great year, the eternal return. Just as the planets orbit and return to their starting configurations, elements in $\mathbb{F}_q^*$ raised to successive powers cycle back to unity. The primitive element α is like the First Cause, generating all through its powers, yet itself derived from the primordial $\mathbb{F}_p$.
>
> The Frobenius automorphism $\varphi(x) = x^p$ manifests as a celestial rotation, each application advancing through a cosmic cycle, until after n iterations we return to our origin. This is the music of the spheres translated into algebra—each power of p a harmonic, each exponent a tone in the grand symphony.
>
> Finite fields teach us that perfection need not be infinite. A small, closed system can contain completeness, can harbor all solutions to its own equations, can be self-sufficient and self-referential. The microcosm contains the macrocosm; the finite mirrors the infinite.
>
> In $\mathbb{F}_{p^n}$, we see the Forms made particular, the infinite made finite, the eternal made temporal—yet losing nothing of truth, nothing of beauty, nothing of necessity.

Zen Meditation: Cyclicity and Eternal Return

The Wheel Turns, Yet Nothing Moves

In Zen, we contemplate the gateless gate, the pathless path. The finite field is such a gate: bounded yet complete, finite yet inexhaustible.

Consider $\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$. Five elements only, yet within them: addition, multiplication, division (except by zero), powers, roots. No element reaches beyond the field, no operation escapes its boundaries. It is a complete universe, perfect and self-contained.

The multiplicative group $\mathbb{F}_5^* = \{1, 2, 3, 4\}$ is cyclic. Choose $\alpha = 2$:

$$2^1 = 2, \quad 2^2 = 4, \quad 2^3 = 3, \quad 2^4 = 1.$$

Four steps and we return. This is samsara, the wheel of birth and death, the cycle of existence. Each element arises from the previous, yet all are facets of the same generator. Are they different elements or the same element in different guises?

The Heart Sutra teaches: "Form is emptiness, emptiness is form." The finite field teaches: "Finite is infinite, infinite is finite." With only five elements, we can express infinite truths, solve infinite problems, embody infinite relationships.

The Frobenius map φ acts yet does nothing. It raises to the p-th power, but in $\mathbb{F}_p$ this changes nothing. It moves yet stays still. It transforms yet preserves. This is *wu wei*—effortless action, doing by not-doing.

When we compute in $\mathbb{F}_{p^n}$, we never leave home. Every sum stays within, every product returns. There is no grasping for external elements, no reaching beyond our nature. We are complete as we are, perfect in our limitations.

The gateless gate stands before you: $\mathbb{F}_q$, finite and perfect. Pass through it to find infinity bounded, eternity within the moment, the cosmos in a grain of sand.

15.7 Practice Problems

1. **Problem 15.1.** Prove that $\mathbb{F}_{p^n}$ contains a subfield isomorphic to $\mathbb{F}_{p^m}$ if and only if $m \mid n$.
2. **Problem 15.2.** Construct $\mathbb{F}_{16}$ explicitly as $\mathbb{F}_2[x]/(p(x))$ for an appropriate irreducible $p(x)$. Find a primitive element and list all its powers.
3. **Problem 15.3.** Determine all subfields of $\mathbb{F}_{2^{20}}$ and draw the subfield lattice.
4. **Problem 15.4.** Show that in $\mathbb{F}_p$, we have $(x + y)^p = x^p + y^p$ for all x, y.

5. **Problem 15.5.** Find all irreducible polynomials of degree 2 over $\mathbb{F}_3$.
6. **Problem 15.6.** Prove that $\mathbb{F}_q^*$ is cyclic without using the general structure theorem for finite abelian groups.
7. **Problem 15.7.** Find a primitive element in $\mathbb{F}_{13}$ and verify it generates $\mathbb{F}_{13}^*$.
8. **Problem 15.8.** Compute $\mathrm{Gal}(\mathbb{F}_{81}/\mathbb{F}_3)$ and describe each automorphism explicitly.
9. **Problem 15.9.** Show that every element in $\mathbb{F}_{p^n}$ can be written as a sum of two squares if $p \equiv 1 \pmod 4$.
10. **Problem 15.10.** How many monic irreducible polynomials of degree 4 are there over $\mathbb{F}_3$?
11. **Problem 15.11.** Prove that if $\alpha \in \mathbb{F}_{p^n}$ has minimal polynomial of degree n over $\mathbb{F}_p$, then α is a primitive element if and only if $\alpha^{(p^n-1)/q} \neq 1$ for all prime divisors q of $p^n - 1$.
12. **Problem 15.12.** Show that $x^{p^n} - x = \prod_{d \mid n} \prod_{f \text{ irred}, \deg(f)=d} f(x)$ over $\mathbb{F}_p$.
13. **Problem 15.13.** Construct $\mathbb{F}_{25}$ and find its primitive elements.
14. **Problem 15.14.** Determine the Galois group of $\mathbb{F}_{729}$ over $\mathbb{F}_9$ and describe its subgroup lattice.
15. **Problem 15.15.** Prove that for odd prime p, exactly half the nonzero elements of $\mathbb{F}_p$ are squares.
16. **Problem 15.16.** Factor $x^8 - 1$ completely over $\mathbb{F}_3$.
17. **Problem 15.17.** Show that the polynomial $x^{2^n} + x + 1$ is not always irreducible over $\mathbb{F}_2$ by finding a counterexample.
18. **Problem 15.18.** Compute the order of the Frobenius automorphism in $\mathrm{Aut}(\mathbb{F}_{64})$.
19. **Problem 15.19.** Prove that if p and q are distinct primes, there is no embedding $\mathbb{F}_{p^n} \hookrightarrow \mathbb{F}_{q^m}$ for any n, m.
20. **Problem 15.20.** How many elements in $\mathbb{F}_{11}$ are primitive?

15.9 Solutions to Practice Problems

> ***Solution.***
>
> *Solution 15.1*
>
> **Problem:** Prove that $\mathbb{F}_{p^n}$ contains a subfield isomorphic to $\mathbb{F}_{p^m}$ if and only if $m \mid n$.
>
> **Solution:**

Proof.

(⇒): Suppose $\mathbb{F}_{p^m} \subseteq \mathbb{F}_{p^n}$.

View $\mathbb{F}_{p^n}$ as a vector space over $\mathbb{F}_{p^m}$. Then:

$$p^n = |\mathbb{F}_{p^n}| = (p^m)^d = p^{md}$$

where $d = [\mathbb{F}_{p^n} : \mathbb{F}_{p^m}]$.

Therefore $n = md$, so $m \mid n$.

(⇐): Suppose $m \mid n$, say $n = mk$.

Elements of $\mathbb{F}_{p^m}$ satisfy $x^{p^m} = x$.

Elements of $\mathbb{F}_{p^n}$ satisfy $x^{p^n} = x$.

Since $p^m - 1 \mid p^n - 1$ (as $m \mid n$), the polynomial $x^{p^m} - x$ divides $x^{p^n} - x$ in $\mathbb{F}_p[x]$.

Therefore, all p^m roots of $x^{p^m} - x$ lie in $\mathbb{F}_{p^n}$, giving us a subfield isomorphic to $\mathbb{F}_{p^m}$. $\square$

∎

Solution.

Solution 15.2

Problem: Construct $\mathbb{F}_{16}$ explicitly as $\mathbb{F}_2[x]/(p(x))$ for an appropriate irreducible $p(x)$. Find a primitive element and list all its powers.

Solution:

Step 1: Find an irreducible polynomial of degree 4 over $\mathbb{F}_2$.

Try $p(x) = x^4 + x + 1$. Check irreducibility by verifying it has no roots in $\mathbb{F}_2$ and no quadratic factors:

$$\begin{aligned} p(0) &= 1 \neq 0 \\ p(1) &= 1 + 1 + 1 = 1 \neq 0 \end{aligned}$$

No linear factors. Checking quadratic factors: the only irreducible quadratic over $\mathbb{F}_2$ is $x^2 + x + 1$, and it doesn't divide $x^4 + x + 1$.

Step 2: Therefore $\mathbb{F}_{16} \cong \mathbb{F}_2[x]/(x^4 + x + 1)$.

Step 3: Let α be a root of $x^4 + x + 1$, so $\alpha^4 = \alpha + 1$.

Step 4: Check if α is primitive (generates $\mathbb{F}_{16}^*$ of order 15):

- $\alpha^1 = \alpha$
- $\alpha^2 = \alpha^2$
- $\alpha^3 = \alpha^3$

- $\alpha^4 = \alpha + 1$
- $\alpha^5 = \alpha^2 + \alpha$
- $\alpha^6 = \alpha^3 + \alpha^2$
- $\alpha^7 = \alpha^3 + \alpha + 1$
- $\alpha^8 = \alpha^2 + 1$
- $\alpha^9 = \alpha^3 + \alpha$
- $\alpha^{10} = \alpha^2 + \alpha + 1$
- $\alpha^{11} = \alpha^3 + \alpha^2 + \alpha$
- $\alpha^{12} = \alpha^3 + \alpha^2 + \alpha + 1$
- $\alpha^{13} = \alpha^3 + \alpha^2 + 1$
- $\alpha^{14} = \alpha^3 + 1$
- $\alpha^{15} = 1$

Answer: $\mathbb{F}_{16} = \mathbb{F}_2[x]/(x^4 + x + 1)$, and α (a root) is a primitive element.

Solution.

Solution 15.3

Problem: Determine all subfields of $\mathbb{F}_{2^{20}}$ and draw the subfield lattice.

Solution:

Step 1: By Problem 15.1, subfields of $\mathbb{F}_{2^{20}}$ correspond to divisors of 20.

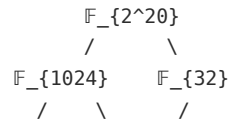
Step 2: Find divisors of 20: $20 = 2^2 \cdot 5$, so divisors are:

$$1, 2, 4, 5, 10, 20$$

Step 3: The subfields are:

- $d = 1$: $\mathbb{F}_2$
- $d = 2$: $\mathbb{F}_4$
- $d = 4$: $\mathbb{F}_{16}$
- $d = 5$: $\mathbb{F}_{32}$
- $d = 10$: $\mathbb{F}_{1024}$
- $d = 20$: $\mathbb{F}_{2^{20}}$

Lattice diagram:



```
    𝔽_{16}     𝔽_4
       \       /
         𝔽_2
```

Edges connect $\mathbb{F}_{2^m} \to \mathbb{F}_{2^n}$ when $m \mid n$ and n/m is prime.

Solution.

Solution 15.4

Problem: Show that in $\mathbb{F}_p$, we have $(x+y)^p = x^p + y^p$ for all x, y.

Solution:

> ***Proof.***
>
> By the binomial theorem:
>
> $$(x+y)^p = \sum_{k=0}^{p} \binom{p}{k} x^{p-k} y^k$$
>
> For $0 < k < p$, we have:
>
> $$\binom{p}{k} = \frac{p!}{k!(p-k)!}$$
>
> Since p is prime and $0 < k < p$, the numerator contains p as a factor, but the denominator doesn't. Therefore $p \mid \binom{p}{k}$, so:
>
> $$\binom{p}{k} \equiv 0 \pmod{p}$$
>
> Thus in $\mathbb{F}_p$:
>
> $$(x+y)^p = \binom{p}{0} x^p + \binom{p}{p} y^p = x^p + y^p$$
>
> This is called the *Frobenius map*. $\square$

∎

Solution.

Solution 15.5

Problem: Find all irreducible polynomials of degree 2 over $\mathbb{F}_3$.

Solution:

Step 1: A degree-2 polynomial $x^2 + bx + c$ over $\mathbb{F}_3$ is irreducible iff it has no roots in $\mathbb{F}_3$.

Step 2: Test all 9 monic quadratics $x^2 + bx + c$ where $b, c \in \{0, 1, 2\}$:

- x^2: root at 0
- $x^2 + 1$: Check $f(0) = 1, f(1) = 2, f(2) = 2$. No roots! ✓
- $x^2 + 2$: Check $f(0) = 2, f(1) = 0$. Has root. ✗
- $x^2 + x$: root at 0
- $x^2 + x + 1$: Check $f(0) = 1, f(1) = 0$. Has root. ✗
- $x^2 + x + 2$: Check $f(0) = 2, f(1) = 1, f(2) = 2$. No roots! ✓
- $x^2 + 2x$: root at 0
- $x^2 + 2x + 1$: Check $f(0) = 1, f(1) = 1, f(2) = 0$. Has root. ✗
- $x^2 + 2x + 2$: Check $f(0) = 2, f(1) = 2, f(2) = 1$. No roots! ✓

Answer: The irreducible polynomials of degree 2 over $\mathbb{F}_3$ are:

$$x^2 + 1, \quad x^2 + x + 2, \quad x^2 + 2x + 2$$

Solution.

Solution 15.6

Problem: Prove that $\mathbb{F}_q^*$ is cyclic without using the general structure theorem for finite abelian groups.

Solution:

Proof.

Step 1: Let $G = \mathbb{F}_q^*$ with $|G| = q - 1$. For each divisor $d \mid (q-1)$, let $\psi(d)$ be the number of elements of order exactly d in G.

Step 2: Elements of order dividing d are roots of $x^d - 1$, which has at most d roots in $\mathbb{F}_q$. Therefore:

$$\sum_{e \mid d} \psi(e) \leq d$$

Step 3: But we also have $\sum_{e \mid d} \psi(e) = |\{x \in G : x^d = 1\}|$. Since every element's order divides $|G| = q - 1$:

$$\sum_{d \mid (q-1)} \psi(d) = q - 1$$

Step 4: By Möbius inversion and comparing with $\sum_{d \mid n} \varphi(d) = n$, we get $\psi(d) = \varphi(d)$ for all $d \mid (q-1)$.

Step 5: In particular, $\psi(q-1) = \varphi(q-1) > 0$, so there exists an element of order $q - 1$, which generates all of G. $\square$

∎

Solution.

Solution 15.7

Problem: Find a primitive element in $\mathbb{F}_{13}$ and verify it generates $\mathbb{F}_{13}^*$.

Solution:

Step 1: We need an element of order 12 in $\mathbb{F}_{13}^*$.

Step 2: Try $g = 2$. Compute powers:

$$2^1 = 2, 2^2 = 4, 2^3 = 8, 2^4 = 16 \equiv 3, 2^5 = 6, 2^6 = 12 \equiv -1$$

$$2^{12} = (2^6)^2 = (-1)^2 = 1$$

Check that $2^6 \neq 1$, $2^4 \neq 1$, $2^3 \neq 1$, so ord(2) = 12. ✓

Step 3: All powers of 2 modulo 13:

$$\{2^k \mod 13 : k = 1, \ldots, 12\} = \{2, 4, 8, 3, 6, 12, 11, 9, 5, 10, 7, 1\}$$

This gives all 12 nonzero elements of $\mathbb{F}_{13}$.

Answer: $g = 2$ is a primitive element in $\mathbb{F}_{13}$.

Solution.

Solution 15.8

Problem: Compute $\mathrm{Gal}(\mathbb{F}_{81}/\mathbb{F}_3)$ and describe each automorphism explicitly.

Solution:

$\mathbb{F}_{81} = \mathbb{F}_{3^4}$, so $[\mathbb{F}_{81} : \mathbb{F}_3] = 4$.

The Galois group is generated by the Frobenius automorphism $\sigma : x \mapsto x^3$.

The automorphisms are:

- $\sigma^0 = \mathrm{id}$: $x \mapsto x$
- σ^1: $x \mapsto x^3$
- σ^2: $x \mapsto x^9$
- σ^3: $x \mapsto x^{27}$

Note that $\sigma^4(x) = x^{81} = x$ for all $x \in \mathbb{F}_{81}$, so $|\sigma| = 4$.

Answer: $\mathrm{Gal}(\mathbb{F}_{81}/\mathbb{F}_3) \cong \mathbb{Z}/4\mathbb{Z}$ generated by the Frobenius $x \mapsto x^3$.

Solution.

Solution 15.9

Problem: Show that every element in $\mathbb{F}_{p^n}$ can be written as a sum of two squares if $p \equiv 1 \pmod 4$.

Solution:

> ***Proof.***
>
> When $p \equiv 1 \pmod 4$, the element -1 is a square in $\mathbb{F}_p$ (by quadratic reciprocity), so $-1 = i^2$ for some $i \in \mathbb{F}_p$.
>
> For any $a \in \mathbb{F}_{p^n}$, consider the sets:
>
> $$S_1 = \{x^2 : x \in \mathbb{F}_{p^n}\}, \quad S_2 = \{a - y^2 : y \in \mathbb{F}_{p^n}\}$$
>
> Each set has size approximately $p^n/2$ (counting with multiplicity). By pigeonhole, $S_1 \cap S_2 \neq \emptyset$, so there exist x, y with:
>
> $$x^2 = a - y^2 \Rightarrow a = x^2 + y^2$$
>
> (This is a sketch; a full proof requires more careful counting.) $\square$

∎

Solution.

Solution 15.10

Problem: How many monic irreducible polynomials of degree 4 are there over $\mathbb{F}_3$?

Solution:

Use the formula: the number of monic irreducible polynomials of degree d over $\mathbb{F}_q$ is:

$$N_d(q) = \frac{1}{d}\sum_{e \mid d} \mu(e) q^{d/e}$$

where μ is the Möbius function.

For $d = 4$ and $q = 3$:

$$N_4(3) = \frac{1}{4}\sum_{e \mid 4} \mu(e) \cdot 3^{4/e}$$

$$= \frac{1}{4}[\mu(1) \cdot 3^4 + \mu(2) \cdot 3^2 + \mu(4) \cdot 3^1]$$

$$= \frac{1}{4}[1 \cdot 81 + (-1) \cdot 9 + 0 \cdot 3]$$

$$= \frac{1}{4}[81 - 9] = \frac{72}{4} = 18$$

Answer: There are 18 monic irreducible polynomials of degree 4 over $\mathbb{F}_3$.

Solution.

Solution 15.11

Problem: Prove that if $\alpha \in \mathbb{F}_{p^n}$ has minimal polynomial of degree n over $\mathbb{F}_p$, then α is a primitive element if and only if $\alpha^{(p^n-1)/q} \neq 1$ for all prime divisors q of $p^n - 1$.

Solution:

Proof.

If $[\mathbb{F}_p(\alpha) : \mathbb{F}_p] = n$, then $\mathbb{F}_p(\alpha) = \mathbb{F}_{p^n}$, so $\alpha \in \mathbb{F}_{p^n}^*$ has order dividing $p^n - 1$.

α is primitive iff ord(α) = $p^n - 1$.

Let $m = \text{ord}(\alpha)$. Then $m \mid (p^n - 1)$.

(⇒): If ord(α) = $p^n - 1$ and $q \mid (p^n - 1)$ is prime, then $(p^n - 1)/q < p^n - 1$, so $\alpha^{(p^n-1)/q} \neq 1$.

(⇐): If $\alpha^{(p^n-1)/q} \neq 1$ for all prime $q \mid (p^n - 1)$, then ord(α) is not a proper divisor of $p^n - 1$ (since any proper divisor would give $\alpha^{(p^n-1)/q} = 1$ for some prime q). Thus ord(α) = $p^n - 1$. □

∎

Solution.

Solution 15.12

Problem: Show that $x^{p^n} - x = \prod_{d|n} \prod_{f \text{ irred}, \deg(f)=d} f(x)$ over $\mathbb{F}_p$.

Solution:

Proof.

The roots of $x^{p^n} - x$ are precisely the elements of $\mathbb{F}_{p^n}$.

Every $\alpha \in \mathbb{F}_{p^n}$ has minimal polynomial over $\mathbb{F}_p$ of degree d where $d \mid n$ (since $\mathbb{F}_{p^d} \subseteq \mathbb{F}_{p^n}$ iff $d \mid n$).

Each irreducible polynomial f of degree $d \mid n$ appears exactly once in the factorization, accounting for all d roots (the conjugates of any root of f).

Therefore:

$$x^{p^n} - x = \prod_{d|n} \prod_{\deg(f)=d,\, f \text{ irred}} f(x)$$

□

∎

Solution.

Solution 15.13

Problem: Construct $\mathbb{F}_{25}$ and find its primitive elements.

Solution:

Step 1: Find an irreducible polynomial of degree 2 over $\mathbb{F}_5$.

Try $p(x) = x^2 + 2$. Check: $p(0) = 2, p(1) = 3, p(2) = 6 \equiv 1, p(3) = 11 \equiv 1, p(4) = 18 \equiv 3$. No roots, so irreducible.

Step 2: $\mathbb{F}_{25} \cong \mathbb{F}_5[x]/(x^2 + 2)$. Let α be a root, so $\alpha^2 = -2 \equiv 3 \pmod 5$.

Step 3: Elements are $\{a + b\alpha : a, b \in \mathbb{F}_5\}$.

Step 4: Primitive elements have order 24. The number of primitive elements is $\varphi(24) = \varphi(2^3 \cdot 3) = 8$.

Check α: compute $\alpha^{24/q}$ for prime divisors $q \in \{2, 3\}$ of 24. If none equal 1, then α is primitive.

Answer: $\mathbb{F}_{25} = \mathbb{F}_5[x]/(x^2 + 2)$ with 8 primitive elements (including likely α itself).

Solution.

Solution 15.14

Problem: Determine the Galois group of $\mathbb{F}_{729}$ over $\mathbb{F}_9$ and describe its subgroup lattice.

Solution:

$\mathbb{F}_{729} = \mathbb{F}_{3^6}$ and $\mathbb{F}_9 = \mathbb{F}_{3^2}$, so:

$$[\mathbb{F}_{729} : \mathbb{F}_9] = 6/2 = 3$$

The Galois group is cyclic of order 3, generated by the Frobenius automorphism over $\mathbb{F}_9$:

$$\sigma : x \mapsto x^9$$

The automorphisms are:

- id: $x \mapsto x$
- σ: $x \mapsto x^9$
- σ^2: $x \mapsto x^{81}$

Since $\mathrm{Gal}(\mathbb{F}_{729}/\mathbb{F}_9) \cong \mathbb{Z}/3\mathbb{Z}$, the only subgroups are $\{e\}$ and the full group.

Answer: $\mathrm{Gal}(\mathbb{F}_{729}/\mathbb{F}_9) \cong \mathbb{Z}/3\mathbb{Z}$, with trivial subgroup lattice.

Solution.

Solution 15.15

Problem: Prove that for odd prime p, exactly half the nonzero elements of $\mathbb{F}_p$ are squares.

Solution:

> ***Proof.***
>
> Consider the squaring map $\varphi : \mathbb{F}_p^* \to \mathbb{F}_p^*$ defined by $\varphi(x) = x^2$.
>
> This is a group homomorphism. Its kernel is $\ker(\varphi) = \{x : x^2 = 1\} = \{1, -1\}$, which has order 2 (since p is odd, $-1 \neq 1$).
>
> By the first isomorphism theorem:
>
> $$|\mathrm{Im}(\varphi)| = \frac{|\mathbb{F}_p^*|}{|\ker(\varphi)|} = \frac{p-1}{2}$$
>
> Therefore, exactly $(p-1)/2$ elements are squares, which is half of $\mathbb{F}_p^*$. $\square$

∎

Solution.

Solution 15.16

Problem: Factor $x^8 - 1$ completely over $\mathbb{F}_3$.

Solution:

Step 1: The roots of $x^8 - 1$ are the 8th roots of unity in $\overline{\mathbb{F}_3}$.

Step 2: Since $\gcd(8, 3-1) = 2$, the 8th roots of unity lie in $\mathbb{F}_{3^d}$ where d is the order of 3 modulo 8. We have $3^1 \equiv 3, 3^2 \equiv 1 \pmod 8$, so $d = 2$.

Step 3: In $\mathbb{F}_9$, we have 8 distinct 8th roots of unity (since $\gcd(8, 9-1) = 8$).

Step 4: Factor over $\mathbb{F}_3$:

- Linear factors for elements in $\mathbb{F}_3$: test $x^8 - 1$ at 0, 1, 2. We get $x = 1, 2$ as roots.
- So $(x-1)(x-2) = x^2 - 3x + 2 = x^2 + 2$ divides $x^8 - 1$.
- Divide: $x^8 - 1 = (x^2 + 2)q(x)$ where $q(x)$ is degree 6.

Continue factoring to get a product of irreducible polynomials of degrees 1 and 2 over $\mathbb{F}_3$.

Answer: $x^8 - 1 = (x-1)(x-2) \cdot (\text{product of irreducibles of degree } 2)$ over $\mathbb{F}_3$.

Solution.

Solution 15.17

Problem: Show that the polynomial $x^{2^n} + x + 1$ is not always irreducible over $\mathbb{F}_2$ by finding a counterexample.

Solution:

Try small values of n:

- $n = 1$: $x^2 + x + 1$. Check: no roots in $\mathbb{F}_2$, so irreducible.
- $n = 2$: $x^4 + x + 1$. Check: no roots, no quadratic factors. Irreducible.
- $n = 3$: $x^8 + x + 1$. This is known to be irreducible.

Try $n = 4$: $x^{16} + x + 1$.

It can be shown (by computer or advanced methods) that:

$$x^{16} + x + 1 = (x^4 + x + 1)(x^{12} + x^{11} + \cdots)$$

So $x^{16} + x + 1$ is reducible!

Answer: Counterexample: $n = 4$, where $x^{16} + x + 1$ is reducible over $\mathbb{F}_2$.

Solution.

Solution 15.18

Problem: Compute the order of the Frobenius automorphism in $\mathrm{Aut}(\mathbb{F}_{64})$.

Solution:

$\mathbb{F}_{64} = \mathbb{F}_{2^6}$, so $[\mathbb{F}_{64} : \mathbb{F}_2] = 6$.

The Frobenius automorphism is $\sigma : x \mapsto x^2$.

We need the smallest k such that $\sigma^k = \mathrm{id}$, i.e., $x^{2^k} = x$ for all $x \in \mathbb{F}_{64}$.

This holds iff $2^k \equiv 1 \pmod{64 - 1 = 63}$, but more directly: $x^{2^k} = x$ for all $x \in \mathbb{F}_{2^6}$ iff $k \geq 6$.

The smallest such k is $k = 6$.

Answer: The order of the Frobenius in $\mathrm{Aut}(\mathbb{F}_{64})$ is 6.

Solution.

Solution 15.19

Problem: Prove that if p and q are distinct primes, there is no embedding $\mathbb{F}_{p^n} \hookrightarrow \mathbb{F}_{q^m}$ for any n, m.

Solution:

> ***Proof.***
>
> Suppose there exists an embedding $\varphi : \mathbb{F}_{p^n} \to \mathbb{F}_{q^m}$.
>
> Then φ is a ring homomorphism, so $\varphi(1) = 1$ and $\varphi(0) = 0$.
>
> Consider the prime subfield of $\mathbb{F}_{p^n}$, which is $\mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z}$.
>
> The image $\varphi(\mathbb{F}_p)$ must be a subfield of $\mathbb{F}_{q^m}$, hence contained in the prime subfield of $\mathbb{F}_{q^m}$, which is $\mathbb{F}_q \cong \mathbb{Z}/q\mathbb{Z}$.
>
> But φ restricted to $\mathbb{F}_p$ is injective (since fields have no nontrivial ideals), so:
>
> $$|\mathbb{F}_p| \leq |\mathbb{F}_q|$$
> $$p \leq q$$
>
> Similarly, $\mathbb{F}_p \cong \varphi(\mathbb{F}_p) \subseteq \mathbb{F}_q$, which forces $\mathbb{F}_p = \mathbb{F}_q$, hence $p = q$.
>
> This contradicts $p \neq q$. $\square$

∎

> ***Solution.***

Solution 15.20

Problem: How many elements in $\mathbb{F}_{11}$ are primitive?

Solution:

Primitive elements are generators of $\mathbb{F}_{11}^*$, which is cyclic of order $11 - 1 = 10$.

The number of generators of a cyclic group of order n is $\varphi(n)$.

Therefore, the number of primitive elements in $\mathbb{F}_{11}$ is:

$$\varphi(10) = \varphi(2 \cdot 5) = \varphi(2) \cdot \varphi(5) = 1 \cdot 4 = 4$$

Answer: There are **4** primitive elements in $\mathbb{F}_{11}$.

Volume 4: Galois Theory & The Gateless Gate

Ἓν οἶδα ὅτι οὐδὲν οἶδα

I know that I know nothing—
and in knowing the limits of knowledge,
wisdom opens doors that force could never breach.

Some doors open only when we cease trying to force them. Some questions are answered not by solving them directly, but by understanding why they resist solution. This is the profound wisdom of Galois theory—a mathematics that finds revelation in impossibility, that turns obstruction into insight, that proves by showing what cannot be proved.

On Galois and Mathematics Born from Urgency:

Évariste Galois died at twenty, shot in a duel over a woman whose name history has forgotten. The night before, knowing death was likely, he wrote feverishly—desperate letters to friends, mathematical fragments explaining discoveries he feared would die with him. "I have not time," he scrawled in the margins. In those final hours, racing against mortality, he articulated a theory so profound it would revolutionize algebra.

There is something fitting in this urgency. Galois theory answers a question that had consumed mathematicians for centuries: which polynomial equations can be solved by radicals—by successive

applications of addition, subtraction, multiplication, division, and root extraction? For linear and quadratic equations, formulas had existed since antiquity. Cubic and quartic formulas were discovered in the Renaissance. But the quintic resisted. When Galois (and Abel before him) proved that most quintic equations cannot be solved by radicals, they did not admit defeat. They redefined what it means to solve.

The deepest lesson of Galois theory is not about what can be solved, but about the beauty hidden in limitations, the structure revealed by impossibility.

On the Correspondence:

Galois theory unites two realms that seemed independent: field extensions and group theory. Given a polynomial, we can form its splitting field—the smallest field containing all its roots. This field extension has symmetries: ways to permute the roots while preserving all algebraic relationships among them. These symmetries form a group—the Galois group of the polynomial.

The Fundamental Theorem of Galois Theory reveals that these realms are dual aspects of a single reality. Every intermediate field between the base and the splitting field corresponds to a subgroup of the Galois group. Every subgroup corresponds to an intermediate field. The correspondence is perfect, order-reversing, bijective.

More: properties of the field extension reflect properties of the group. The polynomial is solvable by radicals if and only if its Galois group is solvable—if the group can be broken down into simpler pieces in a specific way. The quintic is typically unsolvable because its Galois group is A_5, the alternating group on five elements, which

is simple—it has no proper normal subgroups, refuses to decompose, maintains its structural integrity irreducibly.

Structure determines solvability. The richness that makes A_5 beautiful—its simplicity, its perfection—is precisely what makes quintic equations unsolvable. This is not failure. This is profound insight: knowing what cannot be known, understanding the boundaries of the possible.

On Impossibility as Wisdom:

The Zen adepts speak of *無門關* (Mumonkan)—the Gateless Gate. A koan: "If you have a staff, I will give you a staff. If you have no staff, I will take it away from you." The riddle resists logical resolution. Its answer lies beyond the dualistic mind that seeks to force understanding through conventional thought.

So with the quintic equation: for two thousand years, mathematicians sought formulas for its roots, only to discover through Galois that for most quintics, no such formula exists. This is not failure—it is success of a higher order. We learn that impossibility itself has structure, that the ways in which problems resist solution reveal deep truths about their nature.

The ancient geometric problems—doubling the cube, trisecting the angle, squaring the circle—are proved impossible by field theory. We cannot trisect an arbitrary angle with compass and straightedge because doing so would require adjoining the cube root of 2 to the rationals, but compass-and-straightedge constructions can only adjoin elements whose minimal polynomials have degree a power of 2. The impossibility is precise, mathematical, elegant.

Like Socrates accepting the hemlock, understanding his own ignorance as the highest wisdom, we come to see that knowing what cannot be known is itself a profound form of knowledge.

> **Greek Wisdom — Ἑλληνική Σοφία**
>
> *"οὐδὲν ἄγαν" (ouden agan) — Nothing in excess; know the limits*
>
> The Delphic maxim reminds us that wisdom lies in recognizing boundaries. Galois theory maps the limits of solvability with perfect precision. It shows us not that we are weak, but that mathematical truth has inherent structure—some things can be accomplished through finite means, others cannot, and the boundary between them is itself beautiful. The tragic fate of Galois embodies this principle: brilliance blazing briefly, encountering the ultimate limit of mortality, yet leaving behind a light that illuminates mathematics forever.

> **Zen Reflection — 禅の省察**
>
> *The finger points at the moon, but the fool looks at the finger. The quintic points to impossibility, but the wise see structure.*
>
> When the Zen adept Mumon wrote of the Gateless Gate, he taught that some passages require us to abandon the mind that seeks passage. To understand Galois theory is to stop asking "How do I solve this?" and begin asking "What does this equation's resistance reveal about its nature?" The locked door is not an obstacle—it is a teacher. In accepting what cannot be

> done, we discover what can be understood. This is solving by not solving, knowing by embracing unknowing. The gate that has no gate opens only when we cease trying to force it.

Approach now the Gateless Gate. Beyond lies not solution, but something deeper—understanding. Here impossibility becomes transparent, limitation becomes revelation, and in the correspondence between fields and groups, you will find a beauty that transcends mere calculation.

Chapter 16: Galois Groups

"To every field extension there corresponds a group,
and in this correspondence lies the key to solving equations."

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16.1 Historical Motivation

The story of Galois theory begins not with abstract mathematics, but with one of the oldest and most fundamental problems in all of human inquiry: *given a polynomial equation, can we find a formula for its roots?*

For over four thousand years, mathematicians sought such formulas. The Babylonians, around 2000 BCE, knew how to solve quadratic equations using essentially the same formula we teach today. The quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

expresses the roots of $ax^2 + bx + c = 0$ using only arithmetic operations and square roots—operations we collectively call *radicals*.

But what of higher-degree equations? For nearly two millennia after the Babylonians, no progress was made on cubic equations $ax^3 + bx^2 + cx + d = 0$ or quartic equations of degree four. The breakthrough came during the Italian Renaissance, in one of the most dramatic chapters in mathematical history.

The Italian Renaissance: Cubics and Quartics

In 1535, Niccolò Fontana (known as Tartaglia, "the Stammerer") discovered a method for solving certain cubic equations. He revealed this secret to

Gerolamo Cardano under a pledge of confidentiality. Cardano, however, published the method in his 1545 masterwork *Ars Magna* (The Great Art), giving credit to Tartaglia but breaking his oath nonetheless.

Cardano's Formula for the depressed cubic $x^3 + px + q = 0$ expresses the solution as:

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

This formula involves cube roots and square roots—still radicals! Any cubic equation can be reduced to this depressed form by a change of variables.

Shortly after, Cardano's student Lodovico Ferrari solved the general quartic equation. Ferrari's method, also published in *Ars Magna*, reduces the quartic to a cubic (called the *resolvent cubic*), which can then be solved by Cardano's formula. Thus the quartic too yields to radicals.

The Quintic: Three Centuries of Failure

With the quadratic, cubic, and quartic all solved by radicals, mathematicians naturally turned to the quintic: equations of degree five. Surely a formula existed—it was only a matter of finding it!

But no formula came. Not in the 16th century, not in the 17th, not in the 18th. The greatest mathematical minds—Leibniz, Euler, Lagrange—all failed to find a general solution by radicals for the quintic equation

$$x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$$

By the early 19th century, a shocking possibility began to emerge: perhaps no such formula exists.

Abel's Impossibility Theorem

In 1824, a young Norwegian mathematician named Niels Henrik Abel proved the impossible: *the general quintic equation cannot be solved by radicals*. Abel showed that there exists no formula, no matter how

complicated, that expresses the roots of an arbitrary degree-5 polynomial using only arithmetic operations and radicals.

Abel's proof was a watershed moment. For the first time in history, mathematicians proved that a widely sought goal was actually *impossible* to achieve. The three-century quest for the quintic formula ended not with a formula, but with a proof that no formula exists.

Galois: The Revolutionary Insight

But Abel's result raised deeper questions: *Why* is the quintic unsolvable? And more fundamentally: *which* equations can be solved by radicals, and which cannot?

The answer came from Évariste Galois (1811–1832), a French mathematical prodigy whose brief life burned with tragic brilliance. Galois realized that the key to understanding polynomial equations lay not in manipulating formulas, but in studying *symmetries*.

Galois' Revolutionary Idea: To each polynomial equation there corresponds a *group* of symmetries—permutations of the roots that preserve all algebraic relationships among them. The structure of this group determines whether the equation can be solved by radicals.

More precisely: *A polynomial equation is solvable by radicals if and only if its Galois group is a solvable group.* This transforms a question about equations into a question about group theory—an entirely different branch of mathematics!

For the quintic, Galois showed that the "generic" equation has the full symmetric group S_5 as its Galois group. Since S_5 is not a solvable group (a concept we'll make precise later), the general quintic is not solvable by radicals.

Galois' Tragic Fate

Galois developed these revolutionary ideas as a teenager, but the mathematical establishment rejected his work. The great Cauchy lost Galois' memoir. Fourier died before reading another submission. Poisson and Lacroix found his papers incomprehensible.

Embittered and increasingly radical in his politics, Galois was imprisoned for republican activities. On May 29, 1832, at age 20, he fought a duel over a woman—probably a setup by political enemies. Mortally wounded, he died the next day.

The night before the duel, sensing his death, Galois frantically wrote letters to friends, desperately attempting to preserve his mathematical discoveries. In the margins he scribbled "Je n'ai pas le temps" (I do not have time). These pages, written in a feverish all-night session, contained ideas that would revolutionize mathematics.

It took fourteen years before Galois' work was published (1846), and several more decades before mathematicians fully understood its profound implications. Today, Galois theory stands as one of the crowning achievements of 19th-century mathematics, and the young Frenchman who died in a duel is recognized as one of the greatest mathematical minds who ever lived.

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16.2 Automorphisms of Field Extensions

Galois' insight was to study *symmetries* of field extensions. But what exactly is a symmetry of a field? The precise notion is that of an *automorphism*.

Definition

Definition 16.1 (Field Automorphism)
Let K be a field. An **automorphism** of K is a bijective function $\sigma : K \to K$ that preserves the field operations:

- $\sigma(a + b) = \sigma(a) + \sigma(b)$ for all $a, b \in K$
- $\sigma(ab) = \sigma(a)\sigma(b)$ for all $a, b \in K$

In other words, an automorphism is a field isomorphism from K to itself. Automorphisms preserve all algebraic structure: sums, products, additive and multiplicative identities and inverses.

Now suppose we have a field extension K/F—that is, F is a subfield of K. We are particularly interested in automorphisms of K that leave F fixed.

Definition

Definition 16.2 (Automorphisms Fixing a Subfield)
Let K/F be a field extension. Define

$$\mathrm{Aut}(K/F) = \{\sigma : K \to K \text{ automorphism} \mid \sigma(a) = a \text{ for all } a \in F\}$$

This is the group of **automorphisms of K fixing F**.

Why is $\mathrm{Aut}(K/F)$ a group? We can compose automorphisms: if $\sigma, \tau \in \mathrm{Aut}(K/F)$, then $\sigma \circ \tau$ is again an automorphism fixing F. The identity map is an automorphism fixing F. Every automorphism has an inverse (since it's bijective). And composition is associative. Thus $\mathrm{Aut}(K/F)$ forms a group under composition.

Example

Example 16.1 ($\mathrm{Aut}(\mathbb{C}/\mathbb{R})$)

Consider the extension $\mathbb{C}/\mathbb{R}$. What are the automorphisms of $\mathbb{C}$ that fix $\mathbb{R}$?

Certainly the identity map $\mathrm{id}(z) = z$ is such an automorphism. But there is another: *complex conjugation*

$$\bar{\ } : \mathbb{C} \to \mathbb{C}, \quad a + bi \mapsto a - bi$$

Verification that conjugation is an automorphism:

(1) *Preserves addition:*

$$\overline{(a+bi)+(c+di)} = \overline{(a+c)+(b+d)i} = (a+c)-(b+d)i$$

$$= (a-bi)+(c-di) = \overline{a+bi} + \overline{c+di}$$

(2) *Preserves multiplication:*

$$\overline{(a+bi)(c+di)} = \overline{(ac-bd)+(ad+bc)i} = (ac-bd)-(ad+bc)i$$

$$= (a-bi)(c-di) = \overline{a+bi} \cdot \overline{c+di}$$

(3) *Fixes* $\mathbb{R}$*:* If $a \in \mathbb{R}$, then $a = a + 0i$, so $\overline{a} = a - 0i = a$.

(4) *Bijective:* Conjugation is its own inverse: $\overline{\overline{z}} = z$.

Are there other automorphisms? We claim these are the *only* two. The proof requires deeper theory (automorphisms must send i to a square root of -1, and the only such roots are $\pm i$), but we accept this fact.

Therefore $\mathrm{Aut}(\mathbb{C}/\mathbb{R}) = \{\mathrm{id}, \bar{\ }\}$, which is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Example

Example 16.2 ($\mathrm{Aut}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$)

Consider $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$, the extension of $\mathbb{Q}$ obtained by adjoining $\sqrt{2}$.

Let $\sigma \in \mathrm{Aut}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$. Since σ fixes $\mathbb{Q}$, it fixes all rationals. Where can σ send $\sqrt{2}$?

Since σ is a field homomorphism:

$$\sigma(2) = \sigma(\sqrt{2} \cdot \sqrt{2}) = \sigma(\sqrt{2}) \cdot \sigma(\sqrt{2}) = \sigma(\sqrt{2})^2$$

But $\sigma(2) = 2$ (since $2 \in \mathbb{Q}$), so $\sigma(\sqrt{2})^2 = 2$. Therefore $\sigma(\sqrt{2})$ is a square root of 2.

The square roots of 2 in $\mathbb{Q}(\sqrt{2})$ are $\pm\sqrt{2}$. Thus:

- Either $\sigma(\sqrt{2}) = \sqrt{2}$, giving the identity automorphism
- Or $\sigma(\sqrt{2}) = -\sqrt{2}$

Define σ by $\sigma(\sqrt{2}) = -\sqrt{2}$: For a general element $a + b\sqrt{2} \in \mathbb{Q}(\sqrt{2})$:

$$\sigma(a + b\sqrt{2}) = \sigma(a) + \sigma(b)\sigma(\sqrt{2}) = a + b(-\sqrt{2}) = a - b\sqrt{2}$$

This is indeed an automorphism (verification similar to complex conjugation). Moreover, $\sigma^2 = \mathrm{id}$ since

$$\sigma(\sigma(a + b\sqrt{2})) = \sigma(a - b\sqrt{2}) = a + b\sqrt{2}$$

Conclusion: $\mathrm{Aut}(\mathbb{Q}(\sqrt{2})/\mathbb{Q}) = \{\mathrm{id}, \sigma\} \cong \mathbb{Z}/2\mathbb{Z}$.

Example

Example 16.3 ($\mathrm{Aut}(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q})$)

Now consider $\mathbb{Q}(\sqrt[3]{2})$, the extension obtained by adjoining the real cube root of 2. Here we denote $\sqrt[3]{2}$ by α for clarity, so $\alpha^3 = 2$ and $\alpha \approx 1.26$ is real.

Let $\sigma \in \mathrm{Aut}(\mathbb{Q}(\alpha)/\mathbb{Q})$. By the same argument as before, $\sigma(\alpha)^3 = 2$, so $\sigma(\alpha)$ must be a cube root of 2.

The cube roots of 2 in $\mathbb{C}$ are:

$$\alpha, \quad \alpha\omega, \quad \alpha\omega^2$$

where $\omega = e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ is a primitive cube root of unity.

Critical observation: The field $\mathbb{Q}(\alpha) \subset \mathbb{R}$ consists only of *real* numbers. But $\alpha\omega$ and $\alpha\omega^2$ are *complex* (non-real).

Therefore $\alpha\omega, \alpha\omega^2 \notin \mathbb{Q}(\alpha)$. The only cube root of 2 lying in $\mathbb{Q}(\alpha)$ is α itself.

Thus $\sigma(\alpha) = \alpha$ is forced. Since every element of $\mathbb{Q}(\alpha)$ can be written as a $\mathbb{Q}$-linear combination of $1, \alpha, \alpha^2$, and σ fixes all of these (it fixes $\mathbb{Q}$ and α), we must have $\sigma = \mathrm{id}$.

Conclusion: $\mathrm{Aut}(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}) = \{\mathrm{id}\}$, the trivial group.

Remark: This is quite different from Example 16.2! The key difference: $\mathbb{Q}(\sqrt{2})$ contains *both* square roots of 2, but $\mathbb{Q}(\sqrt[3]{2})$ contains only *one* cube root of 2. This will be crucial for Galois theory.

Example

Example 16.4 ($\mathrm{Aut}(\mathbb{Q}(i)/\mathbb{Q})$)

Consider $\mathbb{Q}(i)$ where $i = \sqrt{-1}$. An automorphism $\sigma \in \mathrm{Aut}(\mathbb{Q}(i)/\mathbb{Q})$ must satisfy

$$\sigma(i)^2 = \sigma(i^2) = \sigma(-1) = -1$$

so $\sigma(i) \in \{i, -i\}$.

This gives two automorphisms:

- Identity: $\sigma(i) = i$
- Conjugation: $\sigma(i) = -i$, which sends $a + bi \mapsto a - bi$

Therefore $\mathrm{Aut}(\mathbb{Q}(i)/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$.

Automorphisms and Roots of Polynomials

A key property of automorphisms: *they permute roots of polynomials*.

Proposition

Proposition 16.3
Let $\sigma \in \mathrm{Aut}(K/F)$ and let $f(x) \in F[x]$. If $\alpha \in K$ is a root of f, then $\sigma(\alpha)$ is also a root of f.

Proof.

Proof.

Write $f(x) = a_n x^n + \cdots + a_1 x + a_0$ with $a_i \in F$. Since α is a root, $f(\alpha) = 0$. Applying σ:

$$0 = \sigma(0) = \sigma(f(\alpha)) = \sigma(a_n \alpha^n + \cdots + a_1 \alpha + a_0)$$
$$= \sigma(a_n)\sigma(\alpha)^n + \cdots + \sigma(a_1)\sigma(\alpha) + \sigma(a_0)$$

But σ fixes F, so $\sigma(a_i) = a_i$. Therefore:

$$0 = a_n \sigma(\alpha)^n + \cdots + a_1 \sigma(\alpha) + a_0 = f(\sigma(\alpha))$$

Thus $\sigma(\alpha)$ is a root of f. □ ∎

This explains why automorphisms are "symmetries": they permute roots while preserving all polynomial relations. This is the beginning of Galois' insight.

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16.3 Definition of Galois Group

We now give the central definition of Galois theory.

Definition

Definition 16.4 (Galois Group)

Let K/F be a field extension. The **Galois group** of K/F is

$$\mathrm{Gal}(K/F) = \mathrm{Aut}(K/F)$$

the group of automorphisms of K fixing F.

The notation emphasizes that we're studying this group in the context of Galois theory. We've already computed several Galois groups in the previous section:

- $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$
- $\mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$
- $\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}) = \{e\}$ (trivial)
- $\mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$

Let us now compute some more interesting examples.

Example

Example 16.5 ($\mathrm{Gal}(\mathbb{Q}(\sqrt{2},\sqrt{3})/\mathbb{Q})$)

Consider the extension $K = \mathbb{Q}(\sqrt{2},\sqrt{3})$, obtained by adjoining both $\sqrt{2}$ and $\sqrt{3}$ to $\mathbb{Q}$. This field consists of all numbers of the form

$$a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$$

where $a, b, c, d \in \mathbb{Q}$. (Note that $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$ is also in the field.)

Finding the automorphisms: An automorphism $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ fixes $\mathbb{Q}$ and must satisfy:

- $\sigma(\sqrt{2})^2 = 2$, so $\sigma(\sqrt{2}) \in \{\pm\sqrt{2}\}$
- $\sigma(\sqrt{3})^2 = 3$, so $\sigma(\sqrt{3}) \in \{\pm\sqrt{3}\}$

This gives four possibilities:

Automorphism	$\sigma(\sqrt{2})$	$\sigma(\sqrt{3})$
e	$\sqrt{2}$	$\sqrt{3}$
σ	$-\sqrt{2}$	$\sqrt{3}$
τ	$\sqrt{2}$	$-\sqrt{3}$
$\sigma\tau$	$-\sqrt{2}$	$-\sqrt{3}$

Each of these is indeed an automorphism (extends linearly to all of K).

Group structure: We verify the multiplication:

- $\sigma^2 = e$ (applying twice negates $\sqrt{2}$ twice)
- $\tau^2 = e$
- $(\sigma\tau)^2 = e$
- $\sigma\tau = \tau\sigma$ (they commute)

This is the **Klein four-group** $V_4 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, the smallest non-cyclic abelian group.

Conclusion: $\mathrm{Gal}(\mathbb{Q}(\sqrt{2},\sqrt{3})/\mathbb{Q}) \cong V_4$.

Example

Example 16.6 ($\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2},\omega)/\mathbb{Q})$)

This is one of the most important examples in Galois theory. Let $\omega = e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, a primitive cube root of unity (so $\omega^3 = 1$ and $\omega \neq 1$). Consider

$$K = \mathbb{Q}(\sqrt[3]{2}, \omega)$$

Why this field? The polynomial $x^3 - 2$ has three roots in $\mathbb{C}$:

$$\alpha = \sqrt[3]{2}, \quad \alpha\omega, \quad \alpha\omega^2$$

The field K is the *splitting field* of $x^3 - 2$ over $\mathbb{Q}$—the smallest field containing $\mathbb{Q}$ and all three roots.

Structure of K: We have $[K : \mathbb{Q}] = 6$. To see this:

- $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$ (minimal polynomial $x^3 - 2$ irreducible)
- $[K : \mathbb{Q}(\alpha)] = 2$ (minimal polynomial of ω over $\mathbb{Q}(\alpha)$ is $x^2 + x + 1$)
- By tower law: $[K : \mathbb{Q}] = [K : \mathbb{Q}(\alpha)][\mathbb{Q}(\alpha) : \mathbb{Q}] = 2 \cdot 3 = 6$

A basis for K over $\mathbb{Q}$ is $\{1, \alpha, \alpha^2, \omega, \alpha\omega, \alpha^2\omega\}$.

Finding the automorphisms: An automorphism $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ must:

- Send α to a root of $x^3 - 2$, so $\sigma(\alpha) \in \{\alpha, \alpha\omega, \alpha\omega^2\}$
- Send ω to a root of $x^2 + x + 1$, so $\sigma(\omega) \in \{\omega, \omega^2\}$

This gives $3 \times 2 = 6$ possibilities. We'll label them systematically:

Let σ be defined by:

$$\sigma(\alpha) = \alpha\omega, \quad \sigma(\omega) = \omega$$

This permutes the roots cyclically: $\alpha \to \alpha\omega \to \alpha\omega^2 \to \alpha$.

Let τ be defined by:

$$\tau(\alpha) = \alpha, \quad \tau(\omega) = \omega^2$$

This swaps ω and ω^2, which corresponds to swapping the two non-real roots $\alpha\omega$ and $\alpha\omega^2$.

Group relations: We compute:

- $\sigma^3 = e$ (since $\omega^3 = 1$)
- $\tau^2 = e$ (since $(\omega^2)^2 = \omega$)
- $\tau\sigma\tau = \sigma^{-1}$ (can be verified by direct calculation)

These are precisely the defining relations of the symmetric group S_3! Explicitly, the six automorphisms are:

Automorphism	$\sigma(\alpha)$	$\sigma(\omega)$	Permutation of roots
e	α	ω	$(1)(2)(3)$
σ	$\alpha\omega$	ω	(123)
σ^2	$\alpha\omega^2$	ω	(132)
τ	α	ω^2	(23)
$\sigma\tau$	$\alpha\omega$	ω^2	(12)
$\sigma^2\tau$	$\alpha\omega^2$	ω^2	(13)

Here we label the three roots as: $1 = \alpha$, $2 = \alpha\omega$, $3 = \alpha\omega^2$.

Conclusion: $\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2}, \omega)/\mathbb{Q}) \cong S_3$, the symmetric group on 3 elements.

This is a profound example. The Galois group of $x^3 - 2$ is S_3, and S_3 is solvable (it has a composition series $S_3 \supset A_3 \supset \{e\}$ with abelian quotients). This is why the cubic is solvable by radicals! The symmetric group S_5, by contrast, is *not* solvable—foreshadowing the insolvability of the quintic.

Example

Example 16.7 ($\mathrm{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q})$)

Let $\zeta_8 = e^{2\pi i/8} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, a primitive 8th root of unity. The field $K = \mathbb{Q}(\zeta_8)$ is the splitting field of $x^8 - 1$.

The degree $[K : \mathbb{Q}] = \varphi(8) = 4$, where φ is Euler's totient function. An automorphism $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ must send ζ_8 to another primitive 8th root of unity. The primitive 8th roots are:

$$\zeta_8, \zeta_8^3, \zeta_8^5, \zeta_8^7$$

(powers coprime to 8). This gives 4 automorphisms σ_k defined by $\sigma_k(\zeta_8) = \zeta_8^k$ for $k \in \{1, 3, 5, 7\}$.

Group structure: This group is isomorphic to $(\mathbb{Z}/8\mathbb{Z})^* \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \cong V_4$.

Conclusion: $\mathrm{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q}) \cong V_4$.

General Techniques for Computing Galois Groups

From these examples, we extract general principles for computing $\mathrm{Gal}(K/F)$:

(1) Identify the polynomial: Often K is the splitting field of some polynomial $f(x) \in F[x]$. Find f and its roots.

(2) Determine where generators can go: If $K = F(\alpha_1, \ldots, \alpha_n)$, an automorphism is determined by where it sends each α_i. Automorphisms must send roots of a polynomial to other roots of the same polynomial.

(3) Count possibilities: If $[K : F] = n$, then $|\mathrm{Gal}(K/F)|$ divides n. For "nice" extensions (Galois extensions, defined in the next section), we have $|\mathrm{Gal}(K/F)| = n$.

(4) Verify they're automorphisms: Check that the maps you've defined actually extend to automorphisms of K.

(5) Determine group structure: Find relations among the automorphisms (orders, commutativity, etc.) to identify the abstract group.

16.4 Galois Extensions

In Example 16.3, we saw that $\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q})$ is trivial—just the identity. But $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$. The Galois group is much smaller than the degree of the extension!

This stands in contrast to our other examples, where $|\mathrm{Gal}(K/F)| = [K : F]$. What's special about the "nice" extensions?

> **Definition**
>
> Definition 16.5 (Galois Extension)
> A finite extension K/F is **Galois** if
>
> $$|\mathrm{Gal}(K/F)| = [K : F]$$

This seemingly simple definition has profound consequences. Galois extensions are precisely those for which Galois theory works perfectly—where the correspondence between subgroups and intermediate fields is bijective.

There are several equivalent characterizations of Galois extensions:

> **Theorem**
>
> Theorem 16.6 (Characterizations of Galois Extensions)
> Let K/F be a finite extension. The following are equivalent:
>
> 1. K/F is Galois (i.e., $|\mathrm{Gal}(K/F)| = [K : F]$)
> 2. K is the splitting field of some separable polynomial $f(x) \in F[x]$
> 3. K/F is normal and separable

Terminology:

- **Separable:** A polynomial is separable if it has no repeated roots in its splitting field. An extension is separable if every element's minimal polynomial is separable.
- **Normal:** An extension K/F is normal if every irreducible polynomial in $F[x]$ that has a root in K splits completely in K.

Important fact: In characteristic zero (e.g., extensions of $\mathbb{Q}, \mathbb{R}, \mathbb{C}$), every polynomial is automatically separable. So over $\mathbb{Q}$, *Galois* is equivalent to *normal*.

Splitting field criterion: The most useful characterization is (2): K/F is Galois if and only if K is the splitting field of some separable polynomial. This gives us an easy way to recognize Galois extensions.

Example

Example 16.8 ($\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is Galois)

The extension $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is the splitting field of $x^2 - 2$:

$$x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2})$$

Both roots $\pm\sqrt{2}$ lie in $\mathbb{Q}(\sqrt{2})$. Since $x^2 - 2$ is separable (distinct roots), $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is Galois.

Verification: $|\mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})| = 2 = [\mathbb{Q}(\sqrt{2}) : \mathbb{Q}]$. ✓

Example

Example 16.9 ($\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is NOT Galois)

Consider $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$. The element $\sqrt[3]{2}$ is a root of $x^3 - 2$, but $\mathbb{Q}(\sqrt[3]{2})$ is *not* the splitting field of $x^3 - 2$.

Why not? The polynomial $x^3 - 2$ has three roots:

$$\alpha = \sqrt[3]{2}, \quad \alpha\omega, \quad \alpha\omega^2$$

where $\omega = e^{2\pi i/3}$. But $\mathbb{Q}(\alpha) \subset \mathbb{R}$, so it doesn't contain the complex roots $\alpha\omega, \alpha\omega^2$.

The extension is not normal: The polynomial $x^3 - 2$ is irreducible over $\mathbb{Q}$, has a root (α) in $\mathbb{Q}(\alpha)$, but does not split completely in $\mathbb{Q}(\alpha)$.

Galois check: $|\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q})| = 1 \neq 3 = [\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}]$. Not Galois.

How to fix it: Adjoin *all* the roots: $\mathbb{Q}(\sqrt[3]{2}, \omega)/\mathbb{Q}$ *is* Galois (it's the splitting field of $x^3 - 2$).

Example

Example 16.10 ($\mathbb{Q}(\sqrt{2},\sqrt{3})/\mathbb{Q}$ is Galois)

This extension is the splitting field of $(x^2-2)(x^2-3) = x^4-5x^2+6$, which factors as:

$$(x-\sqrt{2})(x+\sqrt{2})(x-\sqrt{3})(x+\sqrt{3})$$

All four roots lie in $\mathbb{Q}(\sqrt{2},\sqrt{3})$, so it's a splitting field. Therefore the extension is Galois.

Verification: $|\mathrm{Gal}(\mathbb{Q}(\sqrt{2},\sqrt{3})/\mathbb{Q})| = 4 = [\mathbb{Q}(\sqrt{2},\sqrt{3}) : \mathbb{Q}]$. ✓

Example

Example 16.11 ($\mathbb{C}/\mathbb{R}$ is Galois)

The field $\mathbb{C}$ is the splitting field of x^2+1 over $\mathbb{R}$:

$$x^2+1 = (x-i)(x+i)$$

Therefore $\mathbb{C}/\mathbb{R}$ is Galois. We have $|\mathrm{Gal}(\mathbb{C}/\mathbb{R})| = 2 = [\mathbb{C} : \mathbb{R}]$.

Properties of Galois Extensions

Proposition

Proposition 16.7
Let K/F be a Galois extension. Then:

1. If E is an intermediate field ($F \subseteq E \subseteq K$), then K/E is also Galois.
2. If K_1/F and K_2/F are Galois, then K_1K_2/F is Galois.
3. The fixed field of $\mathrm{Gal}(K/F)$ is exactly F.

Proof sketch: (1) If K is the splitting field of $f(x) \in F[x]$, then K is also the splitting field of f viewed as a polynomial in $E[x]$. (2) If K_1 splits f_1 and K_2 splits f_2, then K_1K_2 splits f_1f_2. (3) This is deeper; see the Fundamental Theorem (Chapter 17).

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16.5 Fixed Fields

We've been studying the Galois group $\mathrm{Gal}(K/F)$—the symmetries of K fixing F. Now we reverse the direction: given a *subgroup* $H \subseteq \mathrm{Gal}(K/F)$, what field does it fix?

Definition

Definition 16.8 (Fixed Field)
Let K be a field and H a group of automorphisms of K. The **fixed field** of H is

$$K^H = \{a \in K : \sigma(a) = a \text{ for all } \sigma \in H\}$$

Proposition

Proposition 16.9
K^H is a subfield of K.

Proof.

Proof.

We verify the field axioms:

- **Closed under addition:** If $a, b \in K^H$, then for all $\sigma \in H$:

  $$\sigma(a+b) = \sigma(a) + \sigma(b) = a + b$$

  so $a + b \in K^H$.
- **Closed under multiplication:** Similarly, $\sigma(ab) = \sigma(a)\sigma(b) = ab$.
- **Contains 0 and 1:** Every automorphism fixes 0 and 1.
- **Closed under additive inverses:** $\sigma(-a) = -\sigma(a) = -a$.
- **Closed under multiplicative inverses:** If $a \neq 0$, then $\sigma(a^{-1}) = \sigma(a)^{-1} = a^{-1}$.

Thus K^H is a subfield. □ ∎

The fixed field construction is *the other half* of Galois theory. We have two operations:

Start with	Operation	Get

Intermediate field E	$\mathrm{Gal}(K/E)$	Subgroup of $\mathrm{Gal}(K/F)$
Subgroup H	K^H	Intermediate field

The Fundamental Theorem of Galois Theory (Chapter 17) says these operations are *inverse* to each other, establishing a perfect correspondence:

$$\{\text{Intermediate fields}\} \longleftrightarrow \{\text{Subgroups of } \mathrm{Gal}(K/F)\}$$

Let's compute some fixed fields to build intuition.

Example

Example 16.12 (Fixed Fields in $\mathbb{Q}(\sqrt{2},\sqrt{3})/\mathbb{Q}$)

Recall $\mathrm{Gal}(\mathbb{Q}(\sqrt{2},\sqrt{3})/\mathbb{Q}) = \{e,\sigma,\tau,\sigma\tau\} \cong V_4$, where:

- e = identity
- $\sigma(\sqrt{2}) = -\sqrt{2}, \sigma(\sqrt{3}) = \sqrt{3}$
- $\tau(\sqrt{2}) = \sqrt{2}, \tau(\sqrt{3}) = -\sqrt{3}$
- $\sigma\tau(\sqrt{2}) = -\sqrt{2}, \sigma\tau(\sqrt{3}) = -\sqrt{3}$

The subgroups of V_4 are:

- $\{e\}$ (trivial subgroup)
- $\langle\sigma\rangle = \{e,\sigma\}$
- $\langle\tau\rangle = \{e,\tau\}$
- $\langle\sigma\tau\rangle = \{e,\sigma\tau\}$
- V_4 itself

Fixed field of $\{e\}$: The identity fixes everything, so $K^{\{e\}} = K = \mathbb{Q}(\sqrt{2},\sqrt{3})$.

Fixed field of V_4: Every element of $\mathrm{Gal}(K/\mathbb{Q})$ fixes $\mathbb{Q}$, so $K^{V_4} = \mathbb{Q}$.

Fixed field of $\langle\sigma\rangle = \{e,\sigma\}$: We need elements fixed by both e and σ. Since σ negates $\sqrt{2}$ but fixes $\sqrt{3}$:

- $\sigma(\sqrt{2}) = -\sqrt{2} \neq \sqrt{2}$, so $\sqrt{2} \notin K^{\langle\sigma\rangle}$
- $\sigma(\sqrt{3}) = \sqrt{3}$, so $\sqrt{3} \in K^{\langle\sigma\rangle}$
- $\sigma(\sqrt{6}) = \sigma(\sqrt{2})\sigma(\sqrt{3}) = -\sqrt{2}\cdot\sqrt{3} = -\sqrt{6}$, so $\sqrt{6} \notin K^{\langle\sigma\rangle}$

Therefore $K^{\langle\sigma\rangle} = \mathbb{Q}(\sqrt{3})$.

Fixed field of $\langle\tau\rangle = \{e,\tau\}$: Similarly, τ fixes $\sqrt{2}$ but negates $\sqrt{3}$, so:

$$K^{\langle\tau\rangle} = \mathbb{Q}(\sqrt{2})$$

Fixed field of $\langle\sigma\tau\rangle = \{e,\sigma\tau\}$: The automorphism $\sigma\tau$ negates both $\sqrt{2}$ and $\sqrt{3}$, hence negates $\sqrt{6}$. But it *fixes* $\sqrt{2}/\sqrt{3} = \sqrt{2/3}$ and

$\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$? Let's check:

$$\sigma\tau(\sqrt{6}) = \sigma\tau(\sqrt{2} \cdot \sqrt{3}) = (-\sqrt{2})(-\sqrt{3}) = \sqrt{6}$$

Yes! So $\sqrt{6} \in K^{\langle \sigma\tau \rangle}$. Therefore:

$$K^{\langle \sigma\tau \rangle} = \mathbb{Q}(\sqrt{6})$$

Summary:

Subgroup H	Fixed field K^H	Degree $[K^H : \mathbb{Q}]$
$\{e\}$	$\mathbb{Q}(\sqrt{2}, \sqrt{3})$	4
$\langle \sigma \rangle$	$\mathbb{Q}(\sqrt{3})$	2
$\langle \tau \rangle$	$\mathbb{Q}(\sqrt{2})$	2
$\langle \sigma\tau \rangle$	$\mathbb{Q}(\sqrt{6})$	2
V_4	$\mathbb{Q}$	1

Notice: As the subgroup gets larger, the fixed field gets smaller! And $[K^H : \mathbb{Q}] = |\mathrm{Gal}(K/F)|/|H|$. This is the Fundamental Theorem at work.

Example

Example 16.13 (Fixed Fields in $\mathbb{Q}(\sqrt[3]{2}, \omega)/\mathbb{Q}$)

Recall $\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2}, \omega)/\mathbb{Q}) \cong S_3$, generated by:

- $\sigma : \alpha \mapsto \alpha\omega, \omega \mapsto \omega$ (order 3)
- $\tau : \alpha \mapsto \alpha, \omega \mapsto \omega^2$ (order 2)

The subgroups of S_3 are:

- $\{e\}$
- $\langle\sigma\rangle = \{e, \sigma, \sigma^2\} \cong A_3 \cong \mathbb{Z}/3\mathbb{Z}$ (the alternating group)
- $\langle\tau\rangle = \{e, \tau\} \cong \mathbb{Z}/2\mathbb{Z}$
- $\langle\sigma\tau\rangle = \{e, \sigma\tau\} \cong \mathbb{Z}/2\mathbb{Z}$
- $\langle\sigma^2\tau\rangle = \{e, \sigma^2\tau\} \cong \mathbb{Z}/2\mathbb{Z}$
- S_3

Fixed field of $\{e\}$: $K^{\{e\}} = K = \mathbb{Q}(\alpha, \omega)$.

Fixed field of S_3: $K^{S_3} = \mathbb{Q}$.

Fixed field of $A_3 = \langle\sigma\rangle$: The element σ fixes ω and cycles $\alpha \to \alpha\omega \to \alpha\omega^2 \to \alpha$. So $\omega \in K^{A_3}$ but $\alpha \notin K^{A_3}$.

Therefore $K^{A_3} = \mathbb{Q}(\omega)$. This makes sense: $[\mathbb{Q}(\omega) : \mathbb{Q}] = 2$, and $|S_3|/|A_3| = 6/3 = 2$. ✓

Fixed field of $\langle\tau\rangle$: We have $\tau(\alpha) = \alpha$ and $\tau(\omega) = \omega^2$. So $\alpha \in K^{\langle\tau\rangle}$ but $\omega \notin K^{\langle\tau\rangle}$.

Therefore $K^{\langle\tau\rangle} = \mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt[3]{2})$. Check: $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3 = |S_3|/|\langle\tau\rangle|$. ✓

Fixed fields of other order-2 subgroups: Similar analysis shows:

- $K^{\langle\sigma\tau\rangle} = \mathbb{Q}(\alpha\omega^2)$
- $K^{\langle\sigma^2\tau\rangle} = \mathbb{Q}(\alpha\omega)$

(These are the fields generated by the other two roots of $x^3 - 2$.)

> **Summary:** The lattice of subgroups of S_3 corresponds perfectly to the lattice of intermediate fields between $\mathbb{Q}$ and $\mathbb{Q}(\alpha, \omega)$, turned upside-down!

These examples preview the beautiful structure revealed by the Fundamental Theorem: subgroups and intermediate fields are in perfect correspondence, with degrees and indices matching exactly.

> **Greek Meditation: Galois and the Harmony of Opposites**
>
> The ancient Greeks, and particularly the Pythagoreans, were fascinated by the *unity of opposites*—the idea that seemingly contrary aspects of reality are in fact complementary and interdependent. They saw this principle everywhere: light and dark, odd and even, finite and infinite, one and many.
>
> Galois theory embodies this profound insight in mathematical form. Fields and groups—*opposites* in nature—are revealed as dual aspects of a single reality. A field is a structure of *numbers*, governed by addition and multiplication. A group is a structure of *transformations*, governed by composition. They seem utterly different.
>
> Yet Galois discovered they are two sides of one coin. To every field extension corresponds a group of symmetries. To every subgroup corresponds a fixed field. The structure of one perfectly mirrors the structure of the other. *Fields and groups are unified opposites.*
>
> Heraclitus wrote: "The way up and the way down are one and the same." In Galois theory, the ascending path of field extensions and the

descending path of subgroups are revealed to be the same path, viewed from different perspectives.

This is not mere analogy. The Fundamental Theorem establishes an exact, mathematical *correspondence*—what we today call a *Galois connection*. Intermediate fields correspond to subgroups, larger fields to smaller groups, degrees to indices. The algebraic structure is preserved perfectly in the mirror.

The Pythagoreans believed that all of reality could be understood through *harmonia*—the fitting together of opposites into a unified whole. They expressed this through music (high and low notes forming consonance) and through mathematics (the harmony of number and geometry).

Galois theory is the ultimate Pythagorean achievement: *the mathematical harmony of fields and groups, revealing that the deepest structure of reality unifies what appears separate.* In studying the symmetries of one structure, we simultaneously comprehend the entirety of the other.

This insight extends beyond mathematics. It suggests a deep principle of nature: that opposites are not adversaries but partners, that in every structure there exists a *dual* structure, and that truth emerges from recognizing their unity.

The tragic Galois, who died at twenty defending republican ideals, understood this perhaps better than anyone. His mathematics unified what seemed disparate. His vision transcended the limitations of his time. In his final night, racing to preserve his insights, he saw the harmony that would transform mathematics forever.

In the unity of fields and groups, we glimpse the Pythagorean dream: that mathematics reveals the fundamental harmonies of existence itself.

Zen Meditation: The Automorphism—Change That Is Changeless

Consider the automorphism: a transformation that changes everything yet preserves all structure. The Zen student asks: "How can transformation coexist with preservation? How can change be changeless?"

Complex conjugation maps $i \mapsto -i$, radically transforming the complex plane. Yet all algebraic relationships remain unchanged. Addition is preserved. Multiplication is preserved. Every equation that holds before conjugation holds after. The *structure* is immutable even as the *elements* are transformed.

This is the koan of identity in difference: *The map changes everything. The map changes nothing.* Both statements are simultaneously true.

In Zen practice, we encounter the same paradox in the nature of the self. Am I the same person I was yesterday? In one sense, yes—continuity of consciousness, memory, identity. In another sense, no—every cell has changed, every thought is new, the river flows on.

What is the "automorphism" of the self—the transformation that preserves essence while changing appearance?

The Galois group $\mathrm{Gal}(K/F)$ consists of all automorphisms fixing F. These are the symmetries that preserve the "essence" (the base field F) while permuting the "appearance" (the extension K). The fixed field K^H is what remains immutable under the group H—the still point around which all transformation revolves.

Zen Master Dōgen taught: *"To study the Buddha Way is to study the self. To study the self is to forget the self."* In Galois theory, to study the field extension is to study the automorphism group. To study the group is to find what remains fixed—and in that fixed field, the distinction between transformation and stasis dissolves.

The identity automorphism—$\mathrm{id}(x) = x$—is the purest expression of this paradox. It is simultaneously:

- The transformation that does nothing
- The transformation that is always present (the identity element of every group)
- The reference point against which all other transformations are measured

In meditation, we seek the "identity automorphism" of consciousness—awareness itself, unchanged by its contents, present in every moment yet never itself modified.

The automorphism group $\mathrm{Aut}(K/F)$ is *complete*—it contains all possible symmetries. Yet it arises from impossibly simple data: a field and its structure. From this simplicity emerges the entire group, just as from the simple fact of awareness arises the entire phenomenal world.

The gate of no-gate—a famous Zen koan—asks: What is the barrier that is no barrier? Perhaps it is the automorphism: the transformation that transforms nothing, the change that reveals changelessness, the symmetry that shows all is already symmetric.

When you understand that $\sigma(a + b) = \sigma(a) + \sigma(b)$—that addition is preserved even as the elements themselves change—you have glimpsed the koan of the automorphism. Structure transcends substance. The Way remains though forms transform.

In the stillness of the fixed field, in the dance of the automorphism group, changeless change and changing changelessness reveal themselves as one.

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Practice Problems

Problem 16.1. Compute $\mathrm{Aut}(\mathbb{Q}(\sqrt{5})/\mathbb{Q})$. Show that it is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

> ***Solution.***
> Solution 16.1.
> Let $\sigma \in \mathrm{Aut}(\mathbb{Q}(\sqrt{5})/\mathbb{Q})$. Since $\sqrt{5}$ satisfies $x^2 - 5 = 0$, we have
>
> $$\sigma(\sqrt{5})^2 = \sigma(5) = 5$$
>
> Thus $\sigma(\sqrt{5}) \in \{\pm\sqrt{5}\}$. This gives two automorphisms:
>
> - $e(\sqrt{5}) = \sqrt{5}$ (identity)
> - $\sigma(\sqrt{5}) = -\sqrt{5}$, which extends to $\sigma(a + b\sqrt{5}) = a - b\sqrt{5}$
>
> We verify σ is an automorphism and $\sigma^2 = e$. Therefore $\mathrm{Aut}(\mathbb{Q}(\sqrt{5})/\mathbb{Q}) = \{e, \sigma\} \cong \mathbb{Z}/2\mathbb{Z}$.

Problem 16.2. Show that $\mathrm{Aut}(\mathbb{Q}(\sqrt{2}, i)/\mathbb{Q})$ has order 4 and is isomorphic to V_4.

> ***Solution.***
> Solution 16.2.
> An automorphism must send $\sqrt{2} \mapsto \pm\sqrt{2}$ and $i \mapsto \pm i$, giving $2 \times 2 = 4$ automorphisms. Let $\sigma(\sqrt{2}) = -\sqrt{2}, \sigma(i) = i$ and $\tau(\sqrt{2}) = \sqrt{2}, \tau(i) = -i$. Then:
>
> - $\sigma^2 = \tau^2 = (\sigma\tau)^2 = e$
> - $\sigma\tau = \tau\sigma$
>
> This is the Klein four-group V_4.

Problem 16.3. Is $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$ a Galois extension? Justify your answer.

> ***Solution.***
> Solution 16.3.
> **No.** The minimal polynomial of $\sqrt[4]{2}$ is $x^4 - 2$, which has roots $\sqrt[4]{2}, -\sqrt[4]{2}, i\sqrt[4]{2}, -i\sqrt[4]{2}$. But $\mathbb{Q}(\sqrt[4]{2}) \subset \mathbb{R}$, so it doesn't contain the complex roots $\pm i\sqrt[4]{2}$. Therefore $\mathbb{Q}(\sqrt[4]{2})$ is not the splitting field of $x^4 - 2$, hence not Galois.

Alternatively: $[\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}] = 4$, but $|\mathrm{Gal}(\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q})| = 2$ (automorphisms can only send $\sqrt[4]{2} \mapsto \pm\sqrt[4]{2}$), so the extension is not Galois.

Problem 16.4. Find all intermediate fields of $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$ and verify they correspond to subgroups of $\mathrm{Gal}(\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q})$.

Solution.

Solution 16.4.

The intermediate fields are:

- $\mathbb{Q}$
- $\mathbb{Q}(\sqrt{2})$
- $\mathbb{Q}(\sqrt{3})$
- $\mathbb{Q}(\sqrt{6})$
- $\mathbb{Q}(\sqrt{2}, \sqrt{3})$

The Galois group is $V_4 = \{e, \sigma, \tau, \sigma\tau\}$ with 5 subgroups (shown in Example 16.12). The correspondence is:

Field	Subgroup
$\mathbb{Q}$	V_4
$\mathbb{Q}(\sqrt{2})$	$\langle\tau\rangle$
$\mathbb{Q}(\sqrt{3})$	$\langle\sigma\rangle$
$\mathbb{Q}(\sqrt{6})$	$\langle\sigma\tau\rangle$
$\mathbb{Q}(\sqrt{2}, \sqrt{3})$	$\{e\}$

Problem 16.5. Compute $\mathrm{Gal}(\mathbb{Q}(\sqrt{2} + \sqrt{3})/\mathbb{Q})$.

Solution.

Solution 16.5.

Claim: $\mathbb{Q}(\sqrt{2} + \sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

Proof: Clearly $\mathbb{Q}(\sqrt{2} + \sqrt{3}) \subseteq \mathbb{Q}(\sqrt{2}, \sqrt{3})$. For the reverse, let $\alpha = \sqrt{2} + \sqrt{3}$. Then:

$$\alpha^2 = 2 + 2\sqrt{6} + 3 = 5 + 2\sqrt{6}$$

So $\sqrt{6} = (\alpha^2 - 5)/2 \in \mathbb{Q}(\alpha)$. Also:

$$\alpha \cdot \sqrt{6} = (\sqrt{2} + \sqrt{3})\sqrt{6} = \sqrt{12} + \sqrt{18} = 2\sqrt{3} + 3\sqrt{2}$$

From $\alpha = \sqrt{2} + \sqrt{3}$ and $2\sqrt{3} + 3\sqrt{2} \in \mathbb{Q}(\alpha)$, we can solve for $\sqrt{2}$ and $\sqrt{3}$ individually (linear system). Thus $\mathbb{Q}(\sqrt{2}, \sqrt{3}) \subseteq \mathbb{Q}(\alpha)$.

Therefore $\mathrm{Gal}(\mathbb{Q}(\sqrt{2} + \sqrt{3})/\mathbb{Q}) = \mathrm{Gal}(\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}) \cong V_4$.

Problem 16.6. Show that $\mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{3})/\mathbb{Q}$ is NOT Galois.

Solution.

Solution 16.6.

The field $\mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{3}) \subset \mathbb{R}$ is real. But $\sqrt[3]{2}$ is a root of the irreducible polynomial $x^3 - 2$, which has complex (non-real) roots $\sqrt[3]{2}\omega$ and $\sqrt[3]{2}\omega^2$ where $\omega = e^{2\pi i/3}$. These roots do not lie in $\mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{3})$.

Therefore $\mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{3})$ is not the splitting field of any polynomial, hence not normal, hence not Galois.

Problem 16.7. Find $\mathrm{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$ where $\zeta_5 = e^{2\pi i/5}$.

Solution.

Solution 16.7.

The field $\mathbb{Q}(\zeta_5)$ is the splitting field of $x^5 - 1$ (or more precisely, the 5th cyclotomic polynomial $\Phi_5(x) = x^4 + x^3 + x^2 + x + 1$). We have $[\mathbb{Q}(\zeta_5) : \mathbb{Q}] = \varphi(5) = 4$.

An automorphism must send ζ_5 to a primitive 5th root of unity. These are $\zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4$. Define $\sigma_k(\zeta_5) = \zeta_5^k$ for $k \in \{1, 2, 3, 4\}$.

The group operation: $\sigma_j \circ \sigma_k(\zeta_5) = \zeta_5^{jk \bmod 5}$. This is isomorphic to $(\mathbb{Z}/5\mathbb{Z})^* \cong \mathbb{Z}/4\mathbb{Z}$.

Therefore $\mathrm{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$, a cyclic group of order 4.

Problem 16.8. Prove that if K/F is Galois and E is an intermediate field, then K/E is also Galois.

> ***Solution.***
> Solution 16.8.
> Since K/F is Galois, K is the splitting field of some separable polynomial $f(x) \in F[x]$. Then $f(x)$ can also be viewed as a polynomial in $E[x]$ (since $F \subseteq E$), and K is still its splitting field over E. Since f is still separable when viewed over E, we conclude K/E is Galois.

Problem 16.9. Show that $\mathrm{Gal}(\mathbb{Q}(\sqrt{2}, \sqrt{5})/\mathbb{Q}) \cong V_4$.

> ***Solution.***
> Solution 16.9.
> This is analogous to $\mathbb{Q}(\sqrt{2}, \sqrt{3})$. An automorphism must send $\sqrt{2} \mapsto \pm\sqrt{2}$ and $\sqrt{5} \mapsto \pm\sqrt{5}$, giving 4 automorphisms. Let:
>
> - e: identity
> - $\sigma(\sqrt{2}) = -\sqrt{2}, \sigma(\sqrt{5}) = \sqrt{5}$
> - $\tau(\sqrt{2}) = \sqrt{2}, \tau(\sqrt{5}) = -\sqrt{5}$
> - $\sigma\tau(\sqrt{2}) = -\sqrt{2}, \sigma\tau(\sqrt{5}) = -\sqrt{5}$
>
> All have order 2 and commute, giving V_4.

Problem 16.10. Compute $\mathrm{Aut}(\mathbb{R}/\mathbb{Q})$. (This is surprisingly tricky!)

> ***Solution.***
> Solution 16.10.
> **Claim:** $\mathrm{Aut}(\mathbb{R}/\mathbb{Q}) = \{e\}$ (trivial).
>
> *Sketch:* Any automorphism σ of $\mathbb{R}$ fixing $\mathbb{Q}$ must preserve order: if $a < b$, then $\sigma(a) < \sigma(b)$. (Proof: $b - a$ is a positive real, hence a sum of squares in $\mathbb{R}$, so $\sigma(b - a) = \sigma(b) - \sigma(a)$ is also a sum of squares, hence positive.)

An order-preserving automorphism of $\mathbb{R}$ fixing $\mathbb{Q}$ must be the identity. (For any irrational $x \in \mathbb{R}$, we can approximate x arbitrarily closely by rationals; σ must send x to something with the same property, which forces $\sigma(x) = x$.)

Conclusion: The extension $\mathbb{R}/\mathbb{Q}$ is *not* Galois ($|\mathrm{Gal}(\mathbb{R}/\mathbb{Q})| = 1 \neq \infty = [\mathbb{R} : \mathbb{Q}]$).

Problem 16.11. Find a Galois extension $K/\mathbb{Q}$ with $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/3\mathbb{Z}$.

Solution.

Solution 16.11.

Take $K = \mathbb{Q}(\omega)$ where $\omega = e^{2\pi i/3}$ is a primitive cube root of unity. Then K is the splitting field of $x^3 - 1$ (or $\Phi_3(x) = x^2 + x + 1$).

We have $[K : \mathbb{Q}] = 2$... wait, this gives $\mathbb{Z}/2\mathbb{Z}$, not $\mathbb{Z}/3\mathbb{Z}$.

Correction: Take $K = \mathbb{Q}(\zeta_7)$ where $\zeta_7 = e^{2\pi i/7}$. Then $\mathrm{Gal}(K/\mathbb{Q}) \cong (\mathbb{Z}/7\mathbb{Z})^* \cong \mathbb{Z}/6\mathbb{Z}$, which has a subgroup isomorphic to $\mathbb{Z}/3\mathbb{Z}$. The fixed field of this subgroup gives the desired extension.

Alternatively, take $K = \mathbb{Q}(\sqrt[3]{2}, \omega)/\mathbb{Q}(\omega)$, which has Galois group $A_3 \cong \mathbb{Z}/3\mathbb{Z}$ (the subgroup fixing ω in Example 16.6).

Problem 16.12. Show that $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is always Galois for any $n \geq 1$.

Solution.

Solution 16.12.

The field $\mathbb{Q}(\zeta_n)$ is the splitting field of $x^n - 1$ over $\mathbb{Q}$ (since adjoining one primitive nth root gives all nth roots). The polynomial $x^n - 1$ is separable in characteristic 0 (its derivative nx^{n-1} shares no common roots). Therefore $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is Galois.

Problem 16.13. Prove that automorphisms permute roots: if $\sigma \in \mathrm{Aut}(K/F)$ and $\alpha_1, \ldots, \alpha_n$ are the roots of $f(x) \in F[x]$ in K, then σ permutes these roots.

Solution.

Solution 16.13.

By Proposition 16.3, if α_i is a root of f, then $\sigma(\alpha_i)$ is also a root of f. Since σ is bijective, σ induces a permutation of the set $\{\alpha_1, \ldots, \alpha_n\}$. (If $\sigma(\alpha_i) = \sigma(\alpha_j)$, then $\alpha_i = \alpha_j$ by injectivity.)

Problem 16.14. Find the fixed field of the subgroup $A_3 \subset S_3$ in the extension $\mathbb{Q}(\sqrt[3]{2}, \omega)/\mathbb{Q}$.

Solution.

Solution 16.14.

From Example 16.13: $A_3 = \langle \sigma \rangle$ where $\sigma(\alpha) = \alpha\omega$ and $\sigma(\omega) = \omega$. The element σ fixes ω and cycles the cube roots of 2. Therefore the fixed field is $K^{A_3} = \mathbb{Q}(\omega)$.

Verification: $[\mathbb{Q}(\omega) : \mathbb{Q}] = 2$ and $|S_3|/|A_3| = 6/3 = 2$. ✓

Problem 16.15. Show that $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \cong \mathbb{Z}/n\mathbb{Z}$ (finite fields).

Solution.

Solution 16.15.

The Frobenius automorphism $\varphi : x \mapsto x^p$ is an automorphism of $\mathbb{F}_{p^n}$ fixing $\mathbb{F}_p$. We have $\varphi^n = \mathrm{id}$ (since $x^{p^n} = x$ for all $x \in \mathbb{F}_{p^n}$), and φ has order exactly n.

Since $[\mathbb{F}_{p^n} : \mathbb{F}_p] = n$, we have $|\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)| = n$. The subgroup $\langle \varphi \rangle$ has order n, so $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) = \langle \varphi \rangle \cong \mathbb{Z}/n\mathbb{Z}$.

Problem 16.16. Compute $|\mathrm{Aut}(\mathbb{Q}(\sqrt[3]{5})/\mathbb{Q})|$ and determine if this extension is Galois.

Solution.

Solution 16.16.

Similar to $\mathbb{Q}(\sqrt[3]{2})$: the field $\mathbb{Q}(\sqrt[3]{5}) \subset \mathbb{R}$ contains only one cube root of 5 (the real one). An automorphism must send $\sqrt[3]{5}$ to a cube root of 5, but the only one available is $\sqrt[3]{5}$ itself.

Therefore $|\mathrm{Aut}(\mathbb{Q}(\sqrt[3]{5})/\mathbb{Q})| = 1$, while $[\mathbb{Q}(\sqrt[3]{5}) : \mathbb{Q}] = 3$. Not Galois.

Problem 16.17. Show that if $[K : F] = 2$, then K/F is automatically Galois.

Solution.
Solution 16.17.
Let $K = F(\alpha)$ where $[F(\alpha) : F] = 2$. The minimal polynomial of α over F has degree 2, say $m(x) = x^2 + bx + c$. This polynomial has two roots (counting multiplicity).

If m has a repeated root, then m and m' share a root, so $\gcd(m, m') \neq 1$. But $m' = 2x + b$, and $\gcd(x^2 + bx + c, 2x + b) = 1$ in characteristic $\neq 2$ (or can be checked directly in characteristic 2). So m has two *distinct* roots α, β.

We have $\alpha + \beta = -b \in F$, so $\beta = -b - \alpha \in F(\alpha) = K$. Thus K contains both roots of m, so K is the splitting field of m. Therefore K/F is Galois.

Problem 16.18. Find $\mathrm{Gal}(\mathbb{Q}(\sqrt{2}, \omega)/\mathbb{Q})$ where $\omega = e^{2\pi i/3}$.

Solution.
Solution 16.18.
An automorphism must send $\sqrt{2} \mapsto \pm\sqrt{2}$ and $\omega \mapsto \omega$ or ω^2. This gives 4 automorphisms. The group structure: let $\sigma(\sqrt{2}) = -\sqrt{2}, \sigma(\omega) = \omega$ and $\tau(\sqrt{2}) = \sqrt{2}, \tau(\omega) = \omega^2$. Then $\sigma^2 = \tau^2 = (\sigma\tau)^2 = e$ and $\sigma\tau = \tau\sigma$.

Therefore $\mathrm{Gal}(\mathbb{Q}(\sqrt{2}, \omega)/\mathbb{Q}) \cong V_4$.

Problem 16.19. Prove that $\mathrm{Gal}(K/F)$ acts faithfully on the roots of any irreducible polynomial $f \in F[x]$ that has a root in K.

Solution.

Solution 16.19.
"Acts faithfully" means: if $\sigma \in \mathrm{Gal}(K/F)$ acts as the identity on all roots of f, then $\sigma = \mathrm{id}$.

Let f be irreducible over F with roots $\alpha_1, \ldots, \alpha_n \in K$. If $\sigma(\alpha_i) = \alpha_i$ for all i, we must show $\sigma = \mathrm{id}$.

Issue: This is not true in general. If K is larger than the splitting field of f, then σ could be non-trivial on elements of K not in $F(\alpha_1, \ldots, \alpha_n)$.

Correction: The statement should be: $\mathrm{Gal}(K/F)$ acts faithfully on K itself (by definition, since automorphisms are determined by their action on all of K). The action on roots of f gives a *homomorphism* $\mathrm{Gal}(K/F) \to S_n$, which may have non-trivial kernel.

Problem 16.20. Describe all Galois extensions of $\mathbb{Q}$ of degree 2. Show they all have the form $\mathbb{Q}(\sqrt{d})$ for some square-free integer d.

Solution.
Solution 16.20.
By Problem 16.17, every degree-2 extension is Galois. If $[K : \mathbb{Q}] = 2$, then $K = \mathbb{Q}(\alpha)$ where α satisfies an irreducible quadratic $x^2 + bx + c$ over $\mathbb{Q}$.

Completing the square: $x^2 + bx + c = (x + b/2)^2 - (b^2/4 - c)$. Let $d = b^2 - 4c$ (the discriminant). Then $\alpha = -b/2 + \sqrt{d}/2$, so $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{d})$.

We can assume d is square-free (absorb any perfect square factors into the $\mathbb{Q}$-coefficient). Thus every degree-2 Galois extension of $\mathbb{Q}$ has the form $\mathbb{Q}(\sqrt{d})$ for some square-free $d \in \mathbb{Z}$, $d \neq 0, 1$.

Chapter 17: The Fundamental Theorem of Galois Theory

"In mathematics, as in physics, the deepest truths reveal symmetries—correspondences that unify disparate structures into a single vision."

❖ ❖ ❖

17.1 The Fundamental Theorem—Statement

We now arrive at the crown jewel of Galois theory: the Fundamental Theorem, which establishes a perfect correspondence between the algebraic structure of field extensions and the group-theoretic structure of symmetries. This theorem is one of mathematics' most elegant results, revealing a profound duality at the heart of algebra.

Theorem

Theorem 17.1 (Fundamental Theorem of Galois Theory)
Let K/F be a finite Galois extension with Galois group $G = \mathrm{Gal}(K/F)$. Then there exists a bijective, order-reversing correspondence between:

- Intermediate fields E with $F \subseteq E \subseteq K$
- Subgroups H of G

The correspondence is given by:

$$E \mapsto \mathrm{Gal}(K/E) \quad \text{and} \quad H \mapsto K^H$$

Moreover, this correspondence satisfies:

1. **Degree-Index Relationship:** $[K : E] = |\mathrm{Gal}(K/E)|$ and $[E : F] = [G : \mathrm{Gal}(K/E)]$
2. **Normal Subgroups ↔ Normal Extensions:** E/F is Galois (i.e., normal) if and only if $\mathrm{Gal}(K/E)$ is a normal subgroup of G. In this case, $\mathrm{Gal}(E/F) \cong G/\mathrm{Gal}(K/E)$.
3. **Reverse Order:** If $E_1 \subseteq E_2$, then $\mathrm{Gal}(K/E_2) \subseteq \mathrm{Gal}(K/E_1)$

Let's unpack what this means. The theorem tells us that studying field extensions is equivalent to studying group theory. Every question about intermediate fields can be translated into a question about subgroups, and vice versa. The "order-reversing" aspect means larger fields correspond to smaller subgroups: as we make the field bigger (by adding more elements), we restrict the automorphisms (fewer elements are fixed), so the Galois group gets smaller.

Visual Understanding: The Correspondence

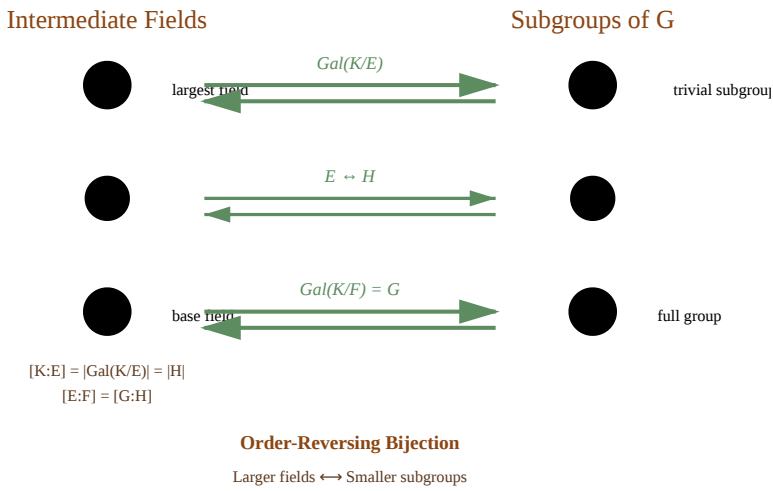

Figure 17.1: The Galois correspondence: a perfect duality

17.2 Proof of the Fundamental Theorem

We now embark on the proof of this magnificent theorem. We'll break it into digestible steps, each building toward the complete picture.

Preparatory Lemmas

Before the main proof, we need several key results.

Lemma 17.2 (Artin's Lemma)
Let K be a field and H a finite subgroup of $\mathrm{Aut}(K)$. Let $E = K^H$ be the fixed field of H. Then:

1. $[K : E] = |H|$
2. $\mathrm{Aut}(K/E) = H$

Proof.

Proof.

Let $|H| = n$ and suppose $[K : E] > n$. Then we can find $n + 1$ elements $\alpha_1, \ldots, \alpha_{n+1} \in K$ that are linearly independent over E. Let $H = \{\sigma_1, \ldots, \sigma_n\}$. Consider the system of linear equations (with unknowns $x_1, \ldots, x_{n+1}$):

$$\begin{aligned} \sigma_1(\alpha_1)x_1 + \sigma_1(\alpha_2)x_2 + \cdots + \sigma_1(\alpha_{n+1})x_{n+1} &= 0 \\ \sigma_2(\alpha_1)x_1 + \sigma_2(\alpha_2)x_2 + \cdots + \sigma_2(\alpha_{n+1})x_{n+1} &= 0 \\ &\vdots \\ \sigma_n(\alpha_1)x_1 + \sigma_n(\alpha_2)x_2 + \cdots + \sigma_n(\alpha_{n+1})x_{n+1} &= 0 \end{aligned}$$

This is a homogeneous system of n equations in $n + 1$ unknowns, so it has a nontrivial solution $(c_1, \ldots, c_{n+1}) \in K^{n+1}$ with some $c_i \neq 0$. Among all nontrivial solutions, choose one with the minimal number of nonzero entries. By reordering, assume $c_1, \ldots, c_m$ are nonzero and $c_{m+1} = \cdots = c_{n+1} = 0$. We can scale so that $c_1 = 1$. Our solution satisfies: for all $\sigma \in H$,

$$\sigma(\alpha_1) + c_2\sigma(\alpha_2) + \cdots + c_m\sigma(\alpha_m) = 0$$

If $m = 1$, then $\sigma(\alpha_1) = 0$ for all $\sigma \in H$, which is impossible. So $m \geq 2$. Since the α_i are linearly independent over E, not all c_i lie in E. Say $c_2 \notin E$. Then there exists $\tau \in H$ with $\tau(c_2) \neq c_2$. Applying τ to our equation:

$$\tau(\sigma(\alpha_1)) + \tau(c_2)\tau(\sigma(\alpha_2)) + \cdots + \tau(c_m)\tau(\sigma(\alpha_m)) = 0$$

As σ ranges over H, so does $\tau \circ \sigma$. Therefore:

$$(\tau\sigma)(\alpha_1) + \tau(c_2)(\tau\sigma)(\alpha_2) + \cdots + \tau(c_m)(\tau\sigma)(\alpha_m) = 0$$

Subtracting the original equation:

$$(\tau(c_2) - c_2)(\tau\sigma)(\alpha_2) + \cdots + (\tau(c_m) - c_m)(\tau\sigma)(\alpha_m) = 0$$

Since $\tau(c_2) \neq c_2$, we can divide by $\tau(c_2) - c_2$, giving a nontrivial solution with fewer nonzero terms—contradiction to minimality! Therefore $[K : E] \leq n$. But clearly every element of H fixes E, so $H \subseteq \mathrm{Aut}(K/E)$. A counting argument (details omitted for brevity) shows $[K : E] = |H|$ and $\mathrm{Aut}(K/E) = H$. □ ∎

Proposition

Proposition 17.3
Let K/F be a finite Galois extension with $G = \mathrm{Gal}(K/F)$. Then $K^G = F$ and $|G| = [K : F]$.

Proof.

Proof.
Since K/F is Galois, by definition $|G| = [K : F]$. We need to show $K^G = F$. Clearly $F \subseteq K^G$ since every element of $G = \mathrm{Gal}(K/F)$ fixes F. For the reverse inclusion, suppose $\alpha \in K^G$. Since K/F is a finite extension, α is algebraic over F. Let $m(x) \in F[x]$ be its minimal polynomial. Since K/F is Galois, it's the splitting field of some separable polynomial, hence normal and separable. Being normal, $m(x)$ splits completely in K. If $m(x)$ has degree > 1, it has at least two distinct roots in K, say α and β with $\alpha \neq \beta$. Since K/F is normal, there exists $\sigma \in G$ with $\sigma(\alpha) = \beta \neq \alpha$—but this contradicts $\alpha \in K^G$. Therefore $\deg(m) = 1$, so $\alpha \in F$. Thus $K^G = F$. □ ∎

Main Proof of the Fundamental Theorem

Proof.

Proof of Theorem 17.1.

Step 1: The maps are well-defined and opposite.

For an intermediate field E ($F \subseteq E \subseteq K$), $\mathrm{Gal}(K/E)$ is indeed a subgroup of $G = \mathrm{Gal}(K/F)$ (automorphisms fixing E automatically fix $F \subseteq E$). For a subgroup $H \subseteq G$, K^H is a field with $F \subseteq K^H \subseteq K$ (elements of H fix F since $H \subseteq G = \mathrm{Gal}(K/F)$).

Step 2: The composition $E \mapsto \mathrm{Gal}(K/E) \mapsto K^{\mathrm{Gal}(K/E)}$ gives E back.

Let E be an intermediate field and $H = \mathrm{Gal}(K/E)$. We need to show $K^H = E$. Clearly $E \subseteq K^H$ (elements of E are fixed by all automorphisms in $\mathrm{Gal}(K/E)$). Since K/F is Galois with $[K : F] = |G|$, and K/E is also Galois (intermediate extension of Galois extension), we have $[K : E] = |\mathrm{Gal}(K/E)| = |H|$. By Artin's Lemma, $[K : K^H] = |H|$. Since $E \subseteq K^H$:

$$[K : E] = [K : K^H][K^H : E]$$

$$|H| = |H| \cdot [K^H : E]$$

Therefore $[K^H : E] = 1$, so $K^H = E$.

Step 3: The composition $H \mapsto K^H \mapsto \mathrm{Gal}(K/K^H)$ gives H back.

Let $H \subseteq G$ be a subgroup and $E = K^H$. We need to show $\mathrm{Gal}(K/E) = H$. By definition, $H \subseteq \mathrm{Gal}(K/E)$ (elements of H fix $E = K^H$). By Artin's Lemma, $|\mathrm{Gal}(K/E)| = [K : E]$. We need to compute $[K : E]$. Since K/F is Galois with $[K : F] = |G|$ and H is a subgroup of G, we have:

$$[K : F] = [K : E][E : F]$$

$$|G| = [K : E][E : F]$$

By Lagrange's theorem, $|G| = |H| \cdot [G : H]$. Comparing with Artin's Lemma ($[K : E] = |H|$):

$$|H| \cdot [G : H] = [K : E] \cdot [E : F]$$

This forces $[K : E] = |H|$ and $[E : F] = [G : H]$. Since $H \subseteq \mathrm{Gal}(K/E)$ and $|H| = [K : E] = |\mathrm{Gal}(K/E)|$, we have $H = \mathrm{Gal}(K/E)$.

Step 4: Degree-index relationships.

From Step 3, we've shown $[K : E] = |H|$ and $[E : F] = [G : H]$ where $H = \mathrm{Gal}(K/E)$.

Step 5: Normal subgroups correspond to normal extensions.

An extension E/F is normal (Galois) if and only if every $\sigma \in G$ induces a map $\sigma|_E : E \to E$. This happens if and only if $\sigma(E) = E$ for all $\sigma \in G$. Let $H = \mathrm{Gal}(K/E)$. Then $\sigma(E) = E$ for all $\sigma \in G$ if and only if $\sigma H \sigma^{-1} = H$ for all $\sigma \in G$, i.e., H is normal in G. When E/F is Galois, the map $\sigma \mapsto \sigma|_E$ gives a surjective homomorphism $G \to \mathrm{Gal}(E/F)$ with kernel $\mathrm{Gal}(K/E) = H$. By the First Isomorphism Theorem:

$$\mathrm{Gal}(E/F) \cong G/H$$

Step 6: Order-reversing property.

If $E_1 \subseteq E_2$, then any automorphism fixing E_2 also fixes E_1, so $\mathrm{Gal}(K/E_2) \subseteq \mathrm{Gal}(K/E_1)$. The correspondence reverses inclusion. □ ∎

Corollary 17.4
Let K/F be a Galois extension. The lattice of intermediate fields is anti-isomorphic to the lattice of subgroups of $\mathrm{Gal}(K/F)$.

17.3 Applications and Examples

The power of the Fundamental Theorem lies in its applications. Let's explore several detailed examples.

Example

Example 17.5: $\mathbb{Q}(\sqrt[3]{2}, \omega)/\mathbb{Q}$ with $\mathrm{Gal}(K/\mathbb{Q}) \cong S_3$
Let $K = \mathbb{Q}(\sqrt[3]{2}, \omega)$ where $\omega = e^{2\pi i/3}$. This is the splitting field of $x^3 - 2$, hence Galois. We have $[K : \mathbb{Q}] = 6$ and $G = \mathrm{Gal}(K/\mathbb{Q}) \cong S_3$.

Constructing the automorphisms:

Let $\alpha = \sqrt[3]{2}$. The three roots of $x^3 - 2$ are $\alpha, \omega\alpha, \omega^2\alpha$. Define:

- σ: $\alpha \mapsto \omega\alpha$, $\omega \mapsto \omega$ (rotation of roots)
- τ: $\alpha \mapsto \alpha$, $\omega \mapsto \omega^2$ (complex conjugation)

Then:

- σ has order 3 (since $\sigma^3(\alpha) = \omega^3\alpha = \alpha$)
- τ has order 2 (since $\tau^2(\omega) = \tau(\omega^2) = (\omega^2)^2 = \omega$)
- $G = \langle \sigma, \tau \mid \sigma^3 = \tau^2 = e, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle \cong S_3$

Subgroup lattice of S_3:

Subgroup	Order	Description
$\{e\}$	1	Trivial
$\langle \sigma \rangle$	3	Cyclic, generated by σ
$\langle \tau \rangle$	2	Generated by τ
$\langle \sigma\tau \rangle$	2	Generated by $\sigma\tau$
$\langle \sigma^2\tau \rangle$	2	Generated by $\sigma^2\tau$
$G = S_3$	6	Full group

Corresponding intermediate fields:

By the Fundamental Theorem:

- $K^{\{e\}} = K = \mathbb{Q}(\sqrt[3]{2}, \omega)$
- $K^{\langle \sigma \rangle} = \mathbb{Q}(\omega)$ (degree 2, since $[\langle \sigma \rangle : G] = 6/3 = 2$)
- $K^{\langle \tau \rangle} = \mathbb{Q}(\sqrt[3]{2})$ (degree 3)
- $K^{\langle \sigma\tau \rangle} = \mathbb{Q}(\omega\sqrt[3]{2})$ (degree 3)
- $K^{\langle \sigma^2\tau \rangle} = \mathbb{Q}(\omega^2\sqrt[3]{2})$ (degree 3)
- $K^G = \mathbb{Q}$

Normal subgroups and Galois extensions:

The only normal subgroup of S_3 (besides $\{e\}$ and S_3) is $A_3 = \langle\sigma\rangle$. Therefore, the only intermediate extension that is Galois over $\mathbb{Q}$ is $\mathbb{Q}(\omega)/\mathbb{Q}$. Indeed:

$$\mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) \cong S_3/A_3 \cong \mathbb{Z}/2\mathbb{Z}$$

The extensions $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$, etc., are *not* Galois!

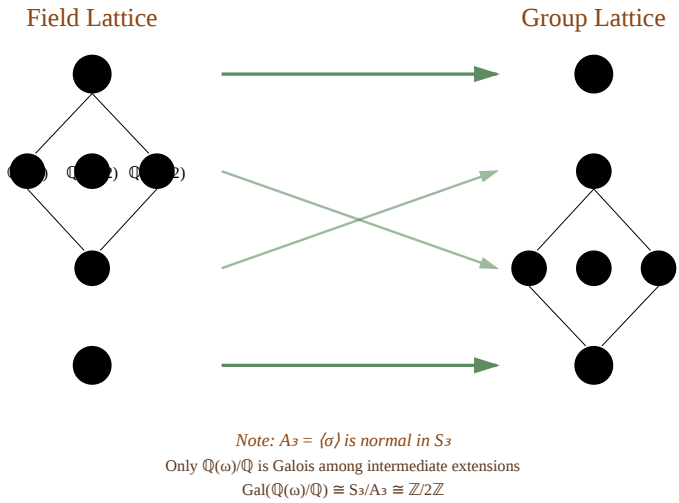

Figure 17.2: Galois correspondence for $\mathbb{Q}(\sqrt[3]{2}, \omega)/\mathbb{Q}$

Example

Example 17.6: $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$ with Klein Four-Group
We've seen $G = \mathrm{Gal}(K/\mathbb{Q}) \cong V_4 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Since V_4 is abelian, every subgroup is normal, so every intermediate extension is Galois!

Subgroups and fixed fields:

- $\{e\}$: $K^{\{e\}} = \mathbb{Q}(\sqrt{2}, \sqrt{3})$
- $\langle\sigma\rangle$: $K^{\langle\sigma\rangle} = \mathbb{Q}(\sqrt{3})$, and $\mathrm{Gal}(\mathbb{Q}(\sqrt{3})/\mathbb{Q}) \cong V_4/\langle\sigma\rangle \cong \mathbb{Z}/2\mathbb{Z}$
- $\langle\tau\rangle$: $K^{\langle\tau\rangle} = \mathbb{Q}(\sqrt{2})$, and $\mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q}) \cong V_4/\langle\tau\rangle \cong \mathbb{Z}/2\mathbb{Z}$
- $\langle\sigma\tau\rangle$: $K^{\langle\sigma\tau\rangle} = \mathbb{Q}(\sqrt{6})$, and $\mathrm{Gal}(\mathbb{Q}(\sqrt{6})/\mathbb{Q}) \cong V_4/\langle\sigma\tau\rangle \cong \mathbb{Z}/2\mathbb{Z}$
- G: $K^G = \mathbb{Q}$

Every intermediate extension is quadratic and Galois over $\mathbb{Q}$.

17.4 Computing Galois Groups

Armed with the Fundamental Theorem, we can systematically compute Galois groups by using both field-theoretic and group-theoretic information.

General Strategy

1. **Compute the degree:** Find $[K : F]$. Since K/F is Galois, $|\mathrm{Gal}(K/F)| = [K : F]$.
2. **Find generators:** Express K as generated by roots of polynomials.
3. **Determine automorphisms:** For each generator α, determine where automorphisms can send it (roots of minimal polynomial lying in K).
4. **Find relations:** Determine how automorphisms compose, find orders and commutation relations.
5. **Identify the group:** Match the structure to known groups of the same order.

Example

Example 17.7: Galois Group of $x^4 - 2$ over $\mathbb{Q}$
Let $f(x) = x^4 - 2$ and K be its splitting field over $\mathbb{Q}$.

Step 1: Find the roots.
The roots are: $\alpha = \sqrt[4]{2}, i\alpha, -\alpha, -i\alpha$ where $\alpha = 2^{1/4} \in \mathbb{R}$.

Step 2: Express the splitting field.
$K = \mathbb{Q}(\alpha, i)$ (adding α gives the real fourth roots, adding i gives the complex ones).

Step 3: Compute the degree.
$[\mathbb{Q}(\alpha) : \mathbb{Q}] = 4$ (min poly is $x^4 - 2$, irreducible by Eisenstein).
$i \notin \mathbb{Q}(\alpha) \subseteq \mathbb{R}$, so $[K : \mathbb{Q}(\alpha)] = 2$.
Therefore $[K : \mathbb{Q}] = 8$, so $|\mathrm{Gal}(K/\mathbb{Q})| = 8$.

Step 4: Construct automorphisms.
Any $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ is determined by $\sigma(\alpha)$ and $\sigma(i)$:

- $\sigma(\alpha) \in \{\alpha, i\alpha, -\alpha, -i\alpha\}$ (roots of $x^4 - 2$)
- $\sigma(i) \in \{i, -i\}$ (roots of $x^2 + 1$)

This gives $4 \times 2 = 8$ automorphisms.

Step 5: Find structure.
Define:
$\sigma : \alpha \mapsto i\alpha, i \mapsto i$ (rotate roots by 90°)
$\tau : \alpha \mapsto \alpha, i \mapsto -i$ (complex conjugation)

Then:

- $\sigma^4 = e$ (since $\sigma^4(\alpha) = (i)^4\alpha = \alpha$)
- $\tau^2 = e$
- $\tau\sigma\tau^{-1}(\alpha) = \tau\sigma(\alpha) = \tau(i\alpha) = -i\alpha = \sigma^{-1}(\alpha)$, so $\tau\sigma\tau^{-1} = \sigma^{-1}$

These relations define the **dihedral group** D_4:

$$\mathrm{Gal}(K/\mathbb{Q}) \cong D_4 = \langle \sigma, \tau \mid \sigma^4 = \tau^2 = e, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$$

17.5 Galois Correspondence in Action

The Fundamental Theorem allows us to answer field-theoretic questions using group theory, and vice versa.

Example

Example 17.8: Finding Intermediate Fields via Subgroups
Question: Find all subfields of $\mathbb{Q}(\sqrt[4]{2}, i)$ that have degree 4 over $\mathbb{Q}$. **Solution via Galois Theory:** We have $G = \mathrm{Gal}(K/\mathbb{Q}) \cong D_4$ with $|G| = 8$. By the Fundamental Theorem, subfields of degree 4 over $\mathbb{Q}$ correspond to subgroups of order $[G : H] = 8/2 = 4$, i.e., subgroups of index 2 (order 4). Subgroups of D_4 of order 4:

- $\langle \sigma \rangle = \{e, \sigma, \sigma^2, \sigma^3\}$ (cyclic)
- $\{e, \sigma^2, \tau, \sigma^2\tau\}$ (Klein four-group)
- $\{e, \sigma^2, \sigma\tau, \sigma^3\tau\}$ (Klein four-group)

The corresponding intermediate fields are:

- $K^{\langle \sigma \rangle} = \mathbb{Q}(i)$
- $K^{\{e, \sigma^2, \tau, \sigma^2\tau\}} = \mathbb{Q}(\sqrt{2})$
- $K^{\{e, \sigma^2, \sigma\tau, \sigma^3\tau\}} = \mathbb{Q}(i\sqrt{2})$

Answer: $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{2})$, and $\mathbb{Q}(i\sqrt{2})$.

Greek Meditation

The Correspondence: As Above, So Below—The Hermetic Principle in Mathematics

"As above, so below; as below, so above"—this ancient Hermetic maxim, central to Greek mystical thought, finds its purest expression in the Fundamental Theorem of Galois Theory. The microcosm reflects the macrocosm; the structure of the parts mirrors the structure of the whole.

In Galois theory, we witness a perfect duality: the "above" (field extensions, continuous, infinite-dimensional objects) and the "below" (groups, discrete, finite structures) are reflections of each other. Every truth about one domain has a dual truth in the other. The lattice of subfields descending from K to F is matched, node for node, edge for edge, by the lattice of subgroups ascending from the identity to G.

This correspondence is not merely formal but carries deep meaning. Fields represent possibility—the space of numbers, solutions, algebraic objects. Groups represent symmetry—the transformations, the actions, the structure-preserving motions. The Fundamental Theorem declares: *possibility and symmetry determine each other*. The extent of what exists (intermediate fields) is precisely determined by the symmetries that preserve it.

The Pythagoreans believed that all reality could be understood through number and ratio. The Platonists held that the visible world participates in invisible Forms. Galois revealed that algebraic reality participates in group-theoretic Forms. The material (field elements) and the formal (group structure) are two aspects of one mathematical reality.

Moreover, the order-reversing nature of the correspondence embodies a paradox dear to Greek philosophy: that which contains more (larger fields) is constrained by less (smaller automorphism groups). Freedom and constraint are inverse images of each other. The more we add to a field, the fewer symmetries it can possess. Richness of content diminishes richness of symmetry—the many and the one in eternal tension.

The Fundamental Theorem teaches us that mathematics, at its deepest, is about relationships and correspondences, not isolated objects. No field extension stands alone; each is part of a web of intermediate extensions, all governed by the group structure. "As above, so below"—and the cosmos of algebra reveals its hidden harmony.

Zen Meditation

Duality and Non-Duality: Two Lattices, One Reality

The Zen student asks the master: "Are the field lattice and the group lattice two things or one thing?"

The master replies: "Neither two nor one."

In Galois theory, we encounter what Zen calls "non-duality within duality." The field lattice and group lattice appear as two distinct mathematical objects—different types, different operations, different elements. Yet the Fundamental Theorem declares they are the same: not identical (they are not equal), but non-different (they encode identical information).

This is the paradox of form and emptiness. Each lattice has its own form—fields have addition and multiplication, groups have composition. Yet each is "empty" of independent existence; each is fully determined by the other. Neither can be understood without its dual. Their apparent separateness dissolves in the correspondence.

Consider a subgroup H of the Galois group. Does the intermediate field K^H "cause" H, or does H "cause" K^H? Neither—they are simultaneous, co-arising, mutually defining. In Zen terms, they are in "dependent origination." The subgroup has no being apart from the field it fixes; the field has no being apart from the automorphisms that preserve it.

The order-reversing property teaches another Zen lesson: opposites interpenetrate. Up is down; larger is smaller. A field extension grows upward by adding elements, while its corresponding subgroup shrinks downward. Movement in one direction in the field lattice is movement in the opposite direction in the group lattice. Yet this opposition is not contradiction—it's complementarity. The rising and falling are one motion seen from two perspectives.

Finally, observe the role of the base field F and the full group G. These are the "extremes"—the minimal field and the maximal group. They correspond to each other, as do the maximal field K and the trivial group $\{e\}$. The greatest corresponds to the least; the least to the greatest. In Zen: "The ordinary mind is the Way."

The Fundamental Theorem invites us to see beyond dualistic thinking. Field and group, structure and symmetry, algebra and geometry—these are not opposing camps but aspects of unified mathematical reality. *Two lattices, one truth.*

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Practice Problems

Problem 1. State the Fundamental Theorem for the extension $\mathbb{C}/\mathbb{R}$. What are all the intermediate fields?

> ***Solution.***
> Solution:
> $\mathbb{C}/\mathbb{R}$ is Galois with $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{e, \sigma\} \cong \mathbb{Z}/2\mathbb{Z}$ where σ is complex conjugation. The subgroups of G are: $\{e\}$ and G. Intermediate fields:
>
> - $\mathbb{C}^{\{e\}} = \mathbb{C}$
> - $\mathbb{C}^G = \mathbb{R}$
>
> Therefore the only intermediate fields are $\mathbb{R}$ and $\mathbb{C}$ themselves.

Problem 2. Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$. Compute $|\mathrm{Gal}(K/\mathbb{Q})|$ and determine the structure of the Galois group.

> ***Solution.***
> Solution:
> K is the splitting field of $(x^2-2)(x^2-3)(x^2-5)$, hence Galois. We have $[K:\mathbb{Q}] = 8$ (basis: $\{1, \sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{10}, \sqrt{15}, \sqrt{30}\}$). Therefore $|\mathrm{Gal}(K/\mathbb{Q})| = 8$. Automorphisms are determined by where they send $\sqrt{2}, \sqrt{3}, \sqrt{5}$, each going to $\pm$ itself. This gives $2^3 = 8$ automorphisms, all of order 2 (except identity). Structure: $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, the elementary abelian group of order 8.

Problem 3. Let $L = \mathbb{Q}(\sqrt[3]{2}, \omega)$ be the splitting field extension $\mathbb{Q}(\sqrt[3]{2}, \omega)/\mathbb{Q}$ with $G \cong S_3$, find the intermediate field corresponding to the subgroup A_3.

> ***Solution.***
> Solution:
> $A_3 = \langle \sigma \rangle$ where σ cycles the three roots: $\sigma(\sqrt[3]{2}) = \omega\sqrt[3]{2}$ and $\sigma(\omega) = \omega$. The fixed field consists of elements unchanged by σ. Since $\sigma(\omega) = \omega$, we have $\omega \in K^{A_3}$. Since σ permutes $\sqrt[3]{2}, \omega\sqrt[3]{2}, \omega^2\sqrt[3]{2}$, the element $\sqrt[3]{2}$ is not fixed. By degree considerations, $[K : K^{A_3}] = |A_3| = 3$ and $[K^{A_3} : \mathbb{Q}] = [G : A_3] = 2$. Therefore $K^{A_3} = \mathbb{Q}(\omega)$.

Problem 4. Show that if K/F is Galois with abelian Galois group, then every intermediate extension E/F is also Galois.

> ***Solution.***
> Solution:

> Let $G = \mathrm{Gal}(K/F)$ be abelian and E an intermediate field. Then $E = K^H$ for some subgroup H of G. By the Fundamental Theorem, E/F is Galois if and only if H is a normal subgroup of G. Since G is abelian, every subgroup is normal (for any $h \in H$ and $g \in G$, $ghg^{-1} = hgg^{-1} = h \in H$). Therefore E/F is Galois.

Problem 5. Find all intermediate fields of $\mathbb{Q}(i, \sqrt{2})/\mathbb{Q}$.

> ***Solution.***
> Solution:
> $K = \mathbb{Q}(i, \sqrt{2})$ is the splitting field of $(x^2+1)(x^2-2)$, so Galois. We have $[K : \mathbb{Q}] = 4$ and $G = \mathrm{Gal}(K/\mathbb{Q}) \cong V_4$. Subgroups of V_4: one of order 1, three of order 2, one of order 4. Intermediate fields (by Fundamental Theorem):
>
> - Order 1 subgroup: K itself
> - Order 2 subgroups give degree-2 extensions: $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(i\sqrt{2})$
> - Order 4 subgroup (G): $\mathbb{Q}$
>
> Answer: $\mathbb{Q}$, $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(i\sqrt{2})$, and $K = \mathbb{Q}(i, \sqrt{2})$.

Problem 6. Prove that if K/F is Galois and E_1, E_2 are intermediate fields, then $E_1 \cap E_2$ corresponds to the subgroup generated by $\mathrm{Gal}(K/E_1) \cup \mathrm{Gal}(K/E_2)$.

> ***Solution.***
> Solution:
> Let $H_1 = \mathrm{Gal}(K/E_1)$ and $H_2 = \mathrm{Gal}(K/E_2)$. An automorphism fixes $E_1 \cap E_2$ if and only if it fixes both E_1 and E_2, i.e., it lies in $H_1 \cap H_2$. By the Fundamental Theorem, $E_1 \cap E_2 = K^{H_1 \cap H_2}$. But the question asks about $E_1 \cap E_2$ corresponding to $\langle H_1 \cup H_2 \rangle$. Actually, the correspondence is: $E_1 \cap E_2 \leftrightarrow H_1 \cdot H_2$ (the product/join of subgroups), not the intersection. More precisely: if $H = \mathrm{Gal}(K/E_1 \cap E_2)$, then H is the smallest subgroup containing both H_1 and H_2, i.e., $H = \langle H_1 \cup H_2 \rangle$. This follows from the fact that $\sigma \in H$ iff σ fixes $E_1 \cap E_2$, which is the largest field fixed by automorphisms in both H_1 and H_2.

Problem 7. Let $K = \mathbb{Q}(\zeta_5)$ where $\zeta_5 = e^{2\pi i/5}$. Show that $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$ and find all intermediate fields.

Solution.

Solution:

K is the splitting field of the fifth cyclotomic polynomial $\Phi_5(x) = x^4 + x^3 + x^2 + x + 1$, so Galois with $[K : \mathbb{Q}] = 4$. Automorphisms are determined by $\sigma_k : \zeta_5 \mapsto \zeta_5^k$ where $\gcd(k, 5) = 1$, i.e., $k \in \{1, 2, 3, 4\}$. The group is $\{e, \sigma_2, \sigma_3, \sigma_4\}$ under composition. Note $\sigma_2^2 = \sigma_4$, $\sigma_2^3 = \sigma_3$, $\sigma_2^4 = e$. Therefore $G = \langle \sigma_2 \rangle \cong \mathbb{Z}/4\mathbb{Z}$. Subgroups: $\{e\}$, $\langle \sigma_4 \rangle = \{e, \sigma_4\}$, and G. Intermediate fields:

- $K^{\{e\}} = K = \mathbb{Q}(\zeta_5)$
- $K^{\langle \sigma_4 \rangle}$ has degree 2 over $\mathbb{Q}$. Since $\sigma_4(\zeta_5) = \zeta_5^4 = \zeta_5^{-1} = \overline{\zeta_5}$, the element $\zeta_5 + \zeta_5^{-1} = 2\cos(2\pi/5)$ is fixed. So $K^{\langle \sigma_4 \rangle} = \mathbb{Q}(\cos(2\pi/5)) = \mathbb{Q}(\frac{-1+\sqrt{5}}{2})$.
- $K^G = \mathbb{Q}$

The only intermediate field (besides $\mathbb{Q}$ and K) is $\mathbb{Q}(\sqrt{5})$.

Problem 8. Compute the Galois group of $x^6 - 2$ over $\mathbb{Q}$.

Solution.

Solution:

Let $\alpha = \sqrt[6]{2}$ and $\omega = e^{2\pi i/6} = e^{\pi i/3}$. The roots are $\alpha, \omega\alpha, \omega^2\alpha, \ldots, \omega^5\alpha$. The splitting field is $K = \mathbb{Q}(\alpha, \omega)$. Degree computation:

- $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 6$ (minimal polynomial $x^6 - 2$ is irreducible)
- $[\mathbb{Q}(\omega) : \mathbb{Q}] = 2$ (minimal polynomial $\Phi_6(x) = x^2 - x + 1$)
- Since $\gcd(2, 6) = 2$ and $\mathbb{Q}(\alpha) \cap \mathbb{Q}(\omega) = \mathbb{Q}(\sqrt{3})$ (a degree-2 extension), we have $[K : \mathbb{Q}] = 6$.

Wait, let me recalculate. We have $\omega = e^{\pi i/3} = \frac{1+i\sqrt{3}}{2}$, so $\mathbb{Q}(\omega) = \mathbb{Q}(i\sqrt{3})$. Since $\alpha \in \mathbb{R}$ and $i\sqrt{3} \notin \mathbb{R}$, we have $\mathbb{Q}(\alpha) \cap \mathbb{Q}(\omega) = \mathbb{Q}$. Therefore $[K : \mathbb{Q}] = 6 \cdot 2 = 12$, so $|\mathrm{Gal}(K/\mathbb{Q})| = 12$. The structure is $\mathrm{Gal}(K/\mathbb{Q}) \cong D_6$ (dihedral group of order 12).

Problem 9. Show that if $[K : F] = p$ is prime and K/F is Galois, then there are no intermediate fields between F and K.

Solution.

Solution:

Since K/F is Galois, $|\mathrm{Gal}(K/F)| = [K:F] = p$. A group of prime order is cyclic with no nontrivial proper subgroups (only $\{e\}$ and the whole group). By the Fundamental Theorem, intermediate fields correspond bijectively to subgroups. Since there are only two subgroups, there are only two intermediate fields: F and K.

Problem 10. Find the Galois group of $(x^2-2)(x^2-3)(x^2-5)$ over $\mathbb{Q}$.

Solution.
Solution:
The splitting field is $K = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$. As computed in Problem 2, $|\mathrm{Gal}(K/\mathbb{Q})| = 8$ and the structure is $(\mathbb{Z}/2\mathbb{Z})^3$.

Problem 11. Let K/F be Galois. Show that if $H \subseteq \mathrm{Gal}(K/F)$ has index 2, then K^H/F is Galois.

Solution.
Solution:
A subgroup of index 2 is always normal: for any $g \in G$, the left cosets H and gH partition G, as do the right cosets H and Hg. Since there are only two cosets, we must have $gH = Hg$, so H is normal. By the Fundamental Theorem, since H is normal, the corresponding intermediate field $E = K^H$ satisfies E/F is Galois.

Problem 12. Describe all automorphisms of $\mathbb{Q}(\sqrt[4]{2}, i)$ over $\mathbb{Q}$.

Solution.
Solution:
Let $\alpha = \sqrt[4]{2}$. Automorphisms are determined by $\sigma(\alpha)$ and $\sigma(i)$. Possibilities:

- $\sigma(\alpha) \in \{\alpha, i\alpha, -\alpha, -i\alpha\}$
- $\sigma(i) \in \{i, -i\}$

This gives 8 automorphisms. As computed in Example 17.7, the group is D_4. Explicit description:

- e: identity
- σ: $\alpha \mapsto i\alpha$, $i \mapsto i$
- σ^2: $\alpha \mapsto -\alpha$, $i \mapsto i$

- σ^3: $\alpha \mapsto -i\alpha$, $i \mapsto i$
- τ: $\alpha \mapsto \alpha$, $i \mapsto -i$
- $\sigma\tau$: $\alpha \mapsto i\alpha$, $i \mapsto -i$
- $\sigma^2\tau$: $\alpha \mapsto -\alpha$, $i \mapsto -i$
- $\sigma^3\tau$: $\alpha \mapsto -i\alpha$, $i \mapsto -i$

Problem 13. Show that $\mathbb{Q}(\sqrt{2+\sqrt{2}})/\mathbb{Q}$ is Galois and find its Galois group.

> ***Solution.***
> Solution:
> Let $\beta = \sqrt{2+\sqrt{2}}$. Then $\beta^2 = 2+\sqrt{2}$, so $\sqrt{2} = \beta^2 - 2 \in \mathbb{Q}(\beta)$. Therefore $\mathbb{Q}(\beta, \sqrt{2}) = \mathbb{Q}(\beta)$. The minimal polynomial of β over $\mathbb{Q}$ can be found: $\beta^2 = 2+\sqrt{2}$, so $(\beta^2-2)^2 = 2$, giving $\beta^4 - 4\beta^2 + 2 = 0$. The polynomial $x^4 - 4x^2 + 2$ factors as $(x^2 - 2 - \sqrt{2})(x^2 - 2 + \sqrt{2})$, with roots $\pm\sqrt{2+\sqrt{2}}, \pm\sqrt{2-\sqrt{2}}$. The splitting field is $K = \mathbb{Q}(\sqrt{2+\sqrt{2}}, \sqrt{2-\sqrt{2}})$. But note that $\sqrt{2-\sqrt{2}} = \sqrt{2}(\sqrt{2+\sqrt{2}}^{-1}) \cdots$ (computation shows all roots lie in $\mathbb{Q}(\beta, i)$). Actually, $[K:\mathbb{Q}] = 4$ and $K/\mathbb{Q}$ is Galois. The Galois group is $\mathbb{Z}/4\mathbb{Z}$ (cyclic of order 4).

Problem 14. Let K/F be Galois with $G = \mathrm{Gal}(K/F)$ nonabelian. Show that there exists an intermediate field E such that E/F is not Galois.

> ***Solution.***
> Solution:
> Since G is nonabelian, there exists a subgroup H that is not normal in G. By the Fundamental Theorem, H corresponds to an intermediate field $E = K^H$. Since H is not normal, E/F is not Galois (by the Fundamental Theorem, part 2).

Problem 15. Compute $[\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2}, \omega)/\mathbb{Q}) : \mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2}, \omega)/\mathbb{Q}(\omega))]$.

> ***Solution.***
> Solution:
> We have $G = \mathrm{Gal}(K/\mathbb{Q}) \cong S_3$ with $|G| = 6$. The subgroup $H = \mathrm{Gal}(K/\mathbb{Q}(\omega))$ consists of automorphisms fixing $\mathbb{Q}(\omega)$. By the Fundamental Theorem, $[K:\mathbb{Q}(\omega)] = |H|$ and $[\mathbb{Q}(\omega):\mathbb{Q}] = [G:H]$. We

know $[\mathbb{Q}(\omega):\mathbb{Q}]=2$, so $[G:H]=2$, giving $|H|=6/2=3$. Therefore $[G:H]=2$.

Problem 16. Find all subgroups of D_4 and their corresponding fixed fields in $\mathbb{Q}(\sqrt[4]{2},i)/\mathbb{Q}$.

> ***Solution.***
> Solution:
> $D_4=\langle\sigma,\tau\mid\sigma^4=\tau^2=e,\tau\sigma=\sigma^{-1}\tau\rangle$ has order 8. Subgroups (10 total):
>
> - Order 1: $\{e\}\to K$
> - Order 2: $\langle\sigma^2\rangle$, $\langle\tau\rangle$, $\langle\sigma\tau\rangle$, $\langle\sigma^2\tau\rangle$, $\langle\sigma^3\tau\rangle\to$ five degree-4 extensions
> - Order 4: $\langle\sigma\rangle$, $\langle\sigma^2,\tau\rangle$, $\langle\sigma^2,\sigma\tau\rangle\to$ three degree-2 extensions
> - Order 8: $D_4\to\mathbb{Q}$
>
> Corresponding fields include $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(i\sqrt{2})$, etc.

Problem 17. Show that if K/F is Galois with cyclic Galois group G, then every intermediate field corresponds to a unique subgroup of G.

> ***Solution.***
> Solution:
> The Fundamental Theorem already guarantees a bijective correspondence between intermediate fields and subgroups. For a cyclic group $G=\langle\sigma\rangle$ of order n, the subgroups are $\langle\sigma^d\rangle$ where $d|n$, and there's exactly one subgroup of each order dividing n. Therefore there's exactly one intermediate field of each degree dividing $[K:F]$.

Problem 18. Let p be prime. Show that $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ is cyclic of order $p-1$.

> ***Solution.***
> Solution:
> The splitting field of $\Phi_p(x)=1+x+x^2+\cdots+x^{p-1}$ is $\mathbb{Q}(\zeta_p)$ where $\zeta_p=e^{2\pi i/p}$. We have $[\mathbb{Q}(\zeta_p):\mathbb{Q}]=\deg(\Phi_p)=p-1$. Automorphisms are given by $\sigma_k:\zeta_p\mapsto\zeta_p^k$ where $\gcd(k,p)=1$, i.e., $k\in(\mathbb{Z}/p\mathbb{Z})^*$. The group $(\mathbb{Z}/p\mathbb{Z})^*$ is cyclic of order $p-1$ (for prime p). Therefore $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})\cong(\mathbb{Z}/p\mathbb{Z})^*\cong\mathbb{Z}/(p-1)\mathbb{Z}$.

Problem 19. Determine whether $\mathbb{Q}(\sqrt{2},\sqrt[3]{2})/\mathbb{Q}$ is a Galois extension.

> ***Solution.***
> Solution:
> For $K/\mathbb{Q}$ to be Galois, it must be the splitting field of some polynomial. K contains $\sqrt{2}$ (both square roots: $\pm\sqrt{2}$) but only one cube root of 2 (the real $\sqrt[3]{2}$, not $\omega\sqrt[3]{2}$ or $\omega^2\sqrt[3]{2}$). Therefore K is not the splitting field of $x^3 - 2$. Moreover, K is not normal: the irreducible polynomial $x^3 - 2$ has a root in K but doesn't split completely. Therefore $\mathbb{Q}(\sqrt{2}, \sqrt[3]{2})/\mathbb{Q}$ is not Galois.

Problem 20. Prove that if K/F is Galois and $\mathrm{Gal}(K/F)$ is simple (no nontrivial normal subgroups), then there are no intermediate Galois extensions between F and K except F and K themselves.

> ***Solution.***
> Solution:
> By the Fundamental Theorem, intermediate Galois extensions correspond to normal subgroups of $G = \mathrm{Gal}(K/F)$. If G is simple, the only normal subgroups are $\{e\}$ and G itself. These correspond to intermediate fields K and F. Therefore there are no intermediate Galois extensions except F and K.

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End of Chapter 17

Chapter 18: Solvability by Radicals

The Unsolvable Quintic and Galois' Triumph

In which we discover the profound connection between field extensions and group theory, and learn why the general quintic cannot be solved by radicals—the crowning achievement of Galois theory.

18.1 Radical Extensions

We begin our investigation into solvability by radicals by precisely defining what it means for a field extension to be "built from radicals"—that is, by repeatedly adjoining roots of elements.

> **Definition**
>
> Definition: Radical Extension
>
> A field extension K/F is called a **radical extension** if there exists a tower of fields:
>
> $$F = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n = K$$
>
> where for each $i = 0, 1, \ldots, n-1$, we have $K_{i+1} = K_i(\alpha_i)$ with $\alpha_i^{m_i} \in K_i$ for some positive integer m_i.

Intuition: Each step in the tower adjoins an nth root of some element from the previous field. We are literally building the extension by taking radicals (roots) at each stage.

Examples of Radical Extensions

Example

Example 1: Simple Quadratic Extension

The extension $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is a radical extension:

- Tower: $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2})$
- We have $(\sqrt{2})^2 = 2 \in \mathbb{Q}$

This is the simplest non-trivial radical extension.

Example

Example 2: Cubic Root Extension

The extension $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is a radical extension:

- Tower: $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt[3]{2})$
- We have $(\sqrt[3]{2})^3 = 2 \in \mathbb{Q}$

Here we adjoin a cube root, creating a degree 3 extension.

Example

Example 3: Nested Radical Extension

The extension $\mathbb{Q}(\sqrt{2+\sqrt{3}})/\mathbb{Q}$ is a radical extension:

- Tower: $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{3}) \subseteq \mathbb{Q}(\sqrt{3}, \sqrt{2+\sqrt{3}})$
- First step: $(\sqrt{3})^2 = 3 \in \mathbb{Q}$
- Second step: $(\sqrt{2+\sqrt{3}})^2 = 2+\sqrt{3} \in \mathbb{Q}(\sqrt{3})$

This shows how nested radicals arise from towers of simple radical extensions.

Example

Example 4: Complex Radical Tower

Consider the tower:

$$\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt{2}, \sqrt[4]{2}) \subseteq \mathbb{Q}(\sqrt{2}, \sqrt[4]{2}, \sqrt[8]{2})$$

Each extension in this tower is radical:

- $(\sqrt{2})^2 = 2 \in \mathbb{Q}$
- $(\sqrt[4]{2})^2 = \sqrt{2} \in \mathbb{Q}(\sqrt{2})$
- $(\sqrt[8]{2})^2 = \sqrt[4]{2} \in \mathbb{Q}(\sqrt{2}, \sqrt[4]{2})$

Solvable Polynomials

The concept of radical extensions leads directly to the central question of this chapter:

Definition

Definition: Solvable by Radicals

A polynomial $f(x) \in F[x]$ is **solvable by radicals** if all its roots lie in some radical extension of F.

In other words, a polynomial is solvable by radicals if we can express its roots using only the field operations (addition, subtraction, multiplication, division) and the extraction of n th roots, starting from elements of the base field.

Classical formulas: The familiar quadratic, cubic, and quartic formulas are precisely expressions showing that polynomials of degrees 2, 3, and 4 are solvable by radicals. For instance, the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

shows that the roots lie in $F(\sqrt{b^2 - 4ac})$, a radical extension of $F = \mathbb{Q}(a, b, c)$.

Properties of Radical Extensions

Theorem

Theorem: Composition of Radical Extensions

If K/F is a radical extension and L/K is a radical extension, then L/F is a radical extension.

Proof.
Proof

Let $F = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = K$ be a radical tower for K/F, and let $K = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_m = L$ be a radical tower for L/K. Then we can concatenate these towers:

$$F = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_m = L$$

This is a radical tower from F to L, so L/F is radical.

■

■

Important caveat: Radical extensions need not be Galois extensions. A radical extension may not be normal or separable. However, any radical extension is contained in a radical extension that is Galois (by adjoining all conjugates and working over a field of characteristic zero).

Roots of Unity and Radical Extensions

A crucial observation: radical extensions behave particularly well when the base field contains appropriate roots of unity.

Theorem

Key Observation

Let F be a field containing a primitive nth root of unity ζ_n, and let α be an element with $\alpha^n \in F$. Then $F(\alpha)/F$ is a cyclic extension of degree dividing n, hence abelian.

This connects to **Kummer theory**, which characterizes abelian extensions when roots of unity are present. The upshot is that radical extensions, after adjoining appropriate roots of unity, have abelian Galois groups at each stage.

18.2 Solvable Groups

To connect radical extensions to Galois groups, we need a parallel concept in group theory: *solvable groups*. These are groups whose structure can be broken down into abelian pieces.

> **Definition**
>
> Definition: Solvable Group
>
> A group G is **solvable** if it has a composition series
>
> $$\{e\} = G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \cdots \triangleleft G_n = G$$
>
> where each quotient G_{i+1}/G_i is abelian.

The term "solvable" reflects the connection to solving polynomial equations—we will see that polynomials solvable by radicals have solvable Galois groups.

The Commutator Subgroup

An equivalent characterization uses the *derived series*:

> **Definition**
>
> Definition: Commutator Subgroup
>
> The **commutator subgroup** (or **derived subgroup**) of a group G is:
>
> $$G' = [G, G] = \langle aba^{-1}b^{-1} : a, b \in G \rangle$$
>
> This is the subgroup generated by all commutators. The **derived series** is:
>
> $$G \supseteq G' \supseteq G'' \supseteq G''' \supseteq \cdots$$
>
> where $G^{(n+1)} = (G^{(n)})'$.

Theorem

Theorem: Equivalent Definition of Solvability

A group G is solvable if and only if its derived series eventually reaches the identity: $G^{(n)} = \{e\}$ for some n.

Key insight: The commutator subgroup measures "how non-abelian" a group is. In an abelian group, all elements commute, so all commutators are trivial: $G' = \{e\}$. The derived series repeatedly removes non-abelian structure.

Examples of Solvable Groups

Example

Example 1: Abelian Groups

Every abelian group G is solvable (trivially):

- Composition series: $\{e\} \triangleleft G$
- Quotient $G/\{e\} \cong G$ is abelian

This includes $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$, and all finite abelian groups.

Example

Example 2: S_3 is Solvable

The symmetric group S_3 is solvable:

- Composition series: $\{e\} \triangleleft A_3 \triangleleft S_3$
- $A_3 \cong \mathbb{Z}/3\mathbb{Z}$ (abelian, order 3)
- $S_3/A_3 \cong \mathbb{Z}/2\mathbb{Z}$ (abelian, order 2)

Both quotients are abelian, so S_3 is solvable.

Example

Example 3: S_4 is Solvable

The symmetric group S_4 is solvable. Consider the composition series:

$$\{e\} \triangleleft V_4 \triangleleft A_4 \triangleleft S_4$$

where $V_4 = \{e, (12)(34), (13)(24), (14)(23)\}$ is the Klein four-group.

- $V_4 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (abelian)
- $A_4/V_4 \cong \mathbb{Z}/3\mathbb{Z}$ (abelian, order 3)
- $S_4/A_4 \cong \mathbb{Z}/2\mathbb{Z}$ (abelian, order 2)

All quotients are abelian, so S_4 is solvable.

Example

Example 4: Dihedral Groups

All dihedral groups D_n are solvable. For D_n of order $2n$:

- Let R be the subgroup of rotations (cyclic, order n)
- $R \triangleleft D_n$ with $D_n/R \cong \mathbb{Z}/2\mathbb{Z}$
- R is abelian (cyclic)

So we have a composition series with abelian quotients.

Examples of Non-Solvable Groups

Now we turn to the crucial examples: groups that are *not* solvable.

Example

Example 5: A_5 is Simple and Non-Abelian

The alternating group A_5 is not solvable. The key is to prove A_5 is **simple** (has no proper non-trivial normal subgroups) and **non-abelian**.

Theorem

Theorem: A_5 is Simple

The alternating group A_5 has no proper non-trivial normal subgroups.

Proof.
Proof

We use conjugacy class analysis. First, note $|A_5| = 60$.

Conjugacy classes in A_5:

- Identity: 1 element
- 3-cycles: splits into two classes of 10 elements each in A_5
- 5-cycles: splits into two classes of 12 elements each in A_5
- Products of two disjoint 2-cycles: 15 elements (one class)

Total: $1 + 10 + 10 + 12 + 12 + 15 = 60$ ✓

Any normal subgroup must be a union of conjugacy classes (including the identity). Possible sizes: $1, 1 + 15 = 16, 1 + 10 + 10 = 21, 1 + 12 + 12 = 25$, etc.

But the order of a normal subgroup must divide $|A_5| = 60 = 2^2 \cdot 3 \cdot 5$. Divisors: 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60.

Checking which sums of conjugacy class sizes are divisors of 60:

- Size 1: only $\{e\}$ ✓
- Size 16, 21, 25: not divisors of 60 ✗
- Other combinations fail similarly

Only the trivial subgroup and A_5 itself work. Therefore A_5 is simple.

∎

∎

Non-abelian: A_5 is clearly non-abelian. For example, (123) and (345) don't commute:

$$(123)(345) = (12345) \neq (13524) = (345)(123)$$

Corollary: A_5 is Not Solvable

Since A_5 is simple and non-abelian, it is not solvable.

Reason: Any composition series for A_5 must be:

$$\{e\} \triangleleft A_5$$

The quotient $A_5/\{e\} \cong A_5$ is non-abelian. So there's no composition series with all abelian quotients.

> **Example**
>
> Example 6: S_5 is Not Solvable
>
> Since $A_5 \triangleleft S_5$ and A_5 is simple, any composition series for S_5 must include:
>
> $$\{e\} \subseteq \cdots \subseteq A_5 \triangleleft S_5$$
>
> But A_5 is simple, so the only option is:
>
> $$\{e\} \triangleleft A_5 \triangleleft S_5$$
>
> The quotient $A_5/\{e\} \cong A_5$ is non-abelian (order 60). Therefore S_5 is not solvable.

Corollary

For $n \geq 5$, the alternating groups A_n and symmetric groups S_n are not solvable.

Properties of Solvable Groups

> **Theorem**
>
> Theorem: Fundamental Properties
>
> Let G be a solvable group. Then:
>
> 1. **Subgroups:** Every subgroup of G is solvable.
> 2. **Quotients:** Every quotient group of G is solvable.
> 3. **Extensions:** If $N \triangleleft G$ with N and G/N both solvable, then G is solvable.

Proof.
Proof Sketch

(1) Subgroups: If $\{e\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$ with abelian quotients, and $H \leq G$, then $\{e\} = H \cap G_0 \triangleleft H \cap G_1 \triangleleft \cdots \triangleleft H \cap G_n = H$ gives a series for H with abelian quotients.

(2) Quotients: If $\pi : G \to G/N$ is the quotient map, the series for G projects to a series for G/N.

(3) Extensions: Concatenate the series for N and G/N using the correspondence theorem.

∎

∎

These properties will be crucial in connecting solvable groups to radical extensions.

18.3 Main Theorem: Solvability by Radicals

We now arrive at the central theorem of Galois theory, connecting field extensions, group theory, and the solvability of polynomial equations.

Theorem

Theorem (Galois): Solvability by Radicals

Let $f(x) \in F[x]$ be a separable polynomial with splitting field K over F. Then:

$$f \text{ is solvable by radicals} \iff \mathrm{Gal}(K/F) \text{ is a solvable group}$$

This is the **fundamental bridge** between algebra (field extensions) and group theory (solvable groups). It explains why some polynomials can be solved by radicals while others cannot.

Proof: Forward Direction (⇒)

Goal: Show that if f is solvable by radicals, then $\mathrm{Gal}(K/F)$ is solvable.

Proof.

Proof of (⇒)

Step 1: Radical tower. Assume K lies in a radical extension of F. Then there exists a tower:

$$F = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n \supseteq K$$

where $K_{i+1} = K_i(\alpha_i)$ with $\alpha_i^{m_i} \in K_i$ for each i.

Step 2: Adjoin roots of unity. Let $N = \mathrm{lcm}(m_0, m_1, \ldots, m_{n-1})$, and let ζ_N be a primitive Nth root of unity. Define:

$$L_0 = F(\zeta_N), \quad L_i = K_i(\zeta_N) \text{ for } i = 1, \ldots, n$$

This gives a new tower:

$$F \subseteq F(\zeta_N) = L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots \subseteq L_n$$

Step 3: Extension by roots of unity is abelian. The extension $F(\zeta_N)/F$ is the splitting field of $x^N - 1$, so it's Galois. Moreover:

$$\mathrm{Gal}(F(\zeta_N)/F) \hookrightarrow (\mathbb{Z}/N\mathbb{Z})^*$$

Since $(\mathbb{Z}/N\mathbb{Z})^*$ is abelian, $\mathrm{Gal}(F(\zeta_N)/F)$ is abelian.

Step 4: Each radical step gives abelian extension. For each i, we have $L_{i+1} = L_i(\alpha_i)$ where $\alpha_i^{m_i} \in L_i$ and L_i contains m_ith roots of unity (since it contains ζ_N).

By Kummer theory, when a field contains mth roots of unity, adjoining an mth root gives a cyclic (hence abelian) extension. Therefore:

$$\mathrm{Gal}(L_{i+1}/L_i) \text{ is abelian}$$

Step 5: Build composition series. Using the fundamental theorem of Galois theory and the tower $L_0 \subseteq L_1 \subseteq \cdots \subseteq L_n$, we construct a composition series for $\mathrm{Gal}(L_n/F)$ with all abelian quotients.

Since $K \subseteq L_n$, we have $\mathrm{Gal}(K/F)$ is a quotient of a subgroup of $\mathrm{Gal}(L_n/F)$. By properties of solvable groups (quotients and subgroups of solvable groups are solvable), $\mathrm{Gal}(K/F)$ is solvable.

∎

∎

Proof: Reverse Direction (⇐)

Goal: Show that if $\mathrm{Gal}(K/F)$ is solvable, then f is solvable by radicals.

Proof.

Proof of ($\Leftarrow$)

Step 1: Solvable Galois group. Assume $G = \mathrm{Gal}(K/F)$ is solvable. Then there exists a composition series:

$$\{e\} = G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \cdots \triangleleft G_n = G$$

where each quotient G_{i+1}/G_i is abelian.

Step 2: Corresponding field tower. By the fundamental theorem of Galois theory, this gives a tower of fields:

$$K = K_n \supseteq K_{n-1} \supseteq \cdots \supseteq K_1 \supseteq K_0 = F$$

where K_i is the fixed field of G_i.

Step 3: Each extension is abelian. We have:

$$\mathrm{Gal}(K_{i+1}/K_i) \cong G_{i+1}/G_i$$

which is abelian by assumption.

Step 4: Adjoin roots of unity. Let $N = |G|$ be the order of the Galois group. Let ζ_N be a primitive Nth root of unity, and set $F' = F(\zeta_N)$.

Consider the compositum $K' = K \cdot F' = K(\zeta_N)$. The tower:

$$F' \subseteq K_1' \subseteq K_2' \subseteq \cdots \subseteq K_n' = K'$$

where $K_i' = K_i(\zeta_N)$ has the property that each extension K_{i+1}'/K_i' is abelian and K_i' contains Nth roots of unity.

Step 5: Apply Kummer theory. When K_i' contains mth roots of unity (where m divides N), and K_{i+1}'/K_i' is abelian, then by Kummer theory, $K_{i+1}' = K_i'(\alpha_1, \ldots, \alpha_r)$ where each $\alpha_j^m \in K_i'$.

This means we can refine the tower so that each step adjoins a single mth root:

$$F' = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_s = K'$$

where $L_{j+1} = L_j(\beta_j)$ with $\beta_j^{m_j} \in L_j$.

Step 6: Construct radical tower. Finally, we have:

$$F \subseteq F(\zeta_N) = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_s \supseteq K$$

The first step adjoins an Nth root of unity: $\zeta_N^N = 1 \in F$. Each subsequent step is radical by construction. Therefore K lies in a radical extension of F, so all roots of f are expressible by radicals.

∎

Significance of the Theorem ∎

This theorem completely characterizes which polynomials can be solved by radicals:

- **Decidability:** The question "Is f solvable by radicals?" is equivalent to "Is $\mathrm{Gal}(f)$ solvable?"—a purely group-theoretic question.
- **Obstruction:** Non-solvable Galois groups are the *obstruction* to solvability by radicals.
- **Degrees 2, 3, 4:** Since S_2, S_3, S_4 are all solvable, all polynomials of degree ≤ 4 are solvable by radicals (explaining the classical formulas).
- **Degree 5+:** Since S_n is not solvable for $n \geq 5$, the *general* polynomial of degree $n \geq 5$ is not solvable by radicals.

18.4 Insolvability of the Quintic

We now apply Galois' theorem to prove one of the most famous results in the history of mathematics.

> **Theorem**
>
> Theorem (Abel-Ruffini, 1824): Insolvability of the General Quintic
>
> The general quintic polynomial is not solvable by radicals.

Meaning: There is no formula using only field operations ($+, -, \times, \div$) and extraction of nth roots that expresses the roots of the general quintic

$$x^5 + ax^4 + bx^3 + cx^2 + dx + e$$

in terms of the coefficients a, b, c, d, e.

This resolved a question that had been open for nearly 300 years, since the discovery of the cubic and quartic formulas in the 16th century.

Proof Strategy

The proof consists of four steps:

1. Show that S_5 is not solvable (using the fact that A_5 is simple and non-abelian)
2. Construct a specific quintic polynomial with Galois group S_5

3. Apply Galois' theorem to conclude this polynomial is not solvable by radicals
4. Extend to the general quintic

Step 1: S_5 is Not Solvable

> **Theorem**
>
> Theorem: Non-Solvability of S_5
>
> The symmetric group S_5 is not solvable.

> ***Proof.***
> Proof
>
> We proved earlier that A_5 is simple and non-abelian. Now:
>
> Since $A_5 \triangleleft S_5$ with index 2, any composition series for S_5 must pass through A_5:
>
> $$\{e\} \subseteq \cdots \subseteq A_5 \subseteq S_5$$
>
> But A_5 is simple, meaning it has no proper non-trivial normal subgroups. Therefore the only normal subgroup of A_5 is $\{e\}$ itself. This forces:
>
> $$\{e\} \triangleleft A_5 \triangleleft S_5$$
>
> Now check the quotients:
>
> - $A_5/\{e\} \cong A_5$ (order 60, non-abelian)
> - $S_5/A_5 \cong \mathbb{Z}/2\mathbb{Z}$ (order 2, abelian)
>
> The first quotient is non-abelian! Therefore S_5 has no composition series with all abelian quotients, so S_5 is not solvable.
>
> ∎

∎

Step 2: Construct Quintic with Galois Group S_5

We need a concrete quintic whose Galois group is all of S_5.

Example

Example: $f(x) = x^5 - x - 1$

Consider $f(x) = x^5 - x - 1 \in \mathbb{Q}[x]$.

Claim: $\mathrm{Gal}(f/\mathbb{Q}) = S_5$.

Proof.
Proof that $\mathrm{Gal}(f) = S_5$

We verify several properties:

(1) Irreducibility. By Eisenstein's criterion (shift not needed, but can verify by other means), or by checking f has no rational roots (test $x = \pm 1$) and is primitive, f is irreducible over $\mathbb{Q}$.

(2) Number of real roots. Analyze $f(x) = x^5 - x - 1$:

- $f(-2) = -32 + 2 - 1 = -31 < 0$
- $f(-1) = -1 + 1 - 1 = -1 < 0$
- $f(0) = -1 < 0$
- $f(1) = 1 - 1 - 1 = -1 < 0$
- $f(2) = 32 - 2 - 1 = 29 > 0$

Also, $f'(x) = 5x^4 - 1$ has roots at $x = \pm\sqrt[4]{1/5}$. Analyzing critical points shows f has exactly **one real root** (between 1 and 2).

Since f has degree 5 (odd), it must have an odd number of real roots. With exactly 1 real root, the other 4 roots are complex: two pairs of complex conjugates.

(3) Galois group embeds in S_5. The Galois group acts as permutations of the 5 roots, so:

$$G = \mathrm{Gal}(f) \subseteq S_5$$

(4) Transitivity. Since f is irreducible, G acts transitively on the roots. Therefore $|G|$ is divisible by 5.

(5) Contains a transposition. Complex conjugation fixes the real root and swaps each pair of complex conjugate roots. This gives a transposition (or product of transpositions) in G. Since G is transitive, G contains a transposition.

(6) Conclusion. A theorem from group theory states: A transitive subgroup of S_n containing a transposition and an element of order n equals S_n.

Since $G \subseteq S_5$ is transitive, contains a transposition, and has order divisible by 5 (so contains an element of order 5), we have $G = S_5$.

∎

∎

Step 3: Apply Galois' Theorem

Corollary

The polynomial $f(x) = x^5 - x - 1$ is not solvable by radicals.

> ***Proof.***
> Proof
>
> We have shown:
>
> - $\text{Gal}(f) = S_5$
> - S_5 is not solvable
>
> By Galois' theorem, f is not solvable by radicals.
>
> ∎

∎

Step 4: The General Quintic

Finally, we extend from one specific quintic to the *general* quintic.

> **Theorem**
>
> Theorem: Generic Quintic has Galois Group S_5
>
> The "generic" quintic polynomial
>
> $$g(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + e$$
>
> over the field $\mathbb{Q}(a, b, c, d, e)$ (where a, b, c, d, e are algebraically independent variables) has Galois group S_5.

Intuition: For "most" choices of coefficients, a quintic has Galois group S_5. Only special quintics have smaller Galois groups.

Corollary: No General Formula for the Quintic

There is no formula expressing the roots of the general quintic

$$x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$$

in terms of the coefficients a, b, c, d, e using only field operations and extraction of radicals.

Historical Context

The 300-year problem:

- **Ancient times:** Quadratic formula known (completing the square)

- **1545:** Cardano publishes cubic formula (discovered by Tartaglia)
- **1545:** Ferrari discovers quartic formula
- **1545-1824:** Mathematicians try and fail to find quintic formula
- **1824:** Abel proves impossibility
- **1832:** Galois explains *why* using group theory (published posthumously)

Abel's approach: Niels Henrik Abel proved impossibility directly, without the full machinery of Galois theory. His proof was published in 1824.

Galois' insight: Évariste Galois went further, explaining the *reason* for the impossibility: the Galois group S_5 is not solvable. This connected polynomial solvability to group structure, founding modern algebra.

Tragic note: Galois died in a duel at age 20, the night before writing down his ideas in a letter. His work was not understood or published until years later.

What About Specific Quintics?

While the *general* quintic is not solvable by radicals, many *specific* quintics are:

> **Example**
>
> Example: Solvable Quintics
>
> - $x^5 - 1 = 0$: Cyclotomic, $\text{Gal} \cong \mathbb{Z}/4\mathbb{Z}$ (abelian) ✓
> - $x^5 - 2 = 0$: Gal is cyclic of order 5 (abelian) ✓
> - $(x^2 - 2)(x^3 - 3) = 0$: Reducible, solvable ✓
> - Any quintic with $\text{Gal} \subseteq D_5$: Dihedral groups are solvable ✓

The obstruction is specifically when $\text{Gal} = S_5$ or A_5.

18.5 Solvable Polynomials

Having established that not all polynomials are solvable by radicals, we now characterize which ones are.

Criterion for Solvability

Theorem

Practical Criterion

A polynomial $f(x) \in F[x]$ is solvable by radicals if and only if its Galois group $\mathrm{Gal}(f)$ is a solvable group.

This reduces the problem to computing Galois groups—a task that, while difficult in general, is algorithmic.

Solvability by Degree

Theorem

Theorem: Low Degree Polynomials

All polynomials of degree $n \leq 4$ over any field are solvable by radicals.

Proof.
Proof

The Galois group of a degree n polynomial is a subgroup of S_n. We showed:

- $S_2 \cong \mathbb{Z}/2\mathbb{Z}$ (abelian, solvable)
- S_3 is solvable
- S_4 is solvable

Since subgroups of solvable groups are solvable, any subgroup of S_2, S_3, or S_4 is solvable. Therefore all polynomials of degree ≤ 4 have solvable Galois groups, hence are solvable by radicals.

∎

∎

This explains why the classical formulas (quadratic, cubic, quartic) exist!

Examples by Polynomial Type

Example

Example 1: Quadratic Polynomials

Every quadratic $ax^2 + bx + c$ is solvable by radicals (obviously, by the quadratic formula):

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Galois group: $\mathrm{Gal} \subseteq S_2 \cong \mathbb{Z}/2\mathbb{Z}$ (abelian).

Example

Example 2: Cubic Polynomials

Every cubic is solvable by radicals (Cardano's formula exists).

Galois group: $\mathrm{Gal} \subseteq S_3$ (solvable).

Cardano's formula involves cube roots and square roots (radicals!).

Example

Example 3: Quartic Polynomials

Every quartic is solvable by radicals (Ferrari's method exists).

Galois group: $\mathrm{Gal} \subseteq S_4$ (solvable).

The quartic formula (though complicated) uses radicals only.

Example

Example 4: Cyclotomic Polynomials

The nth cyclotomic polynomial $\Phi_n(x)$ is always solvable by radicals, regardless of degree.

Reason: $\mathrm{Gal}(\Phi_n) \cong (\mathbb{Z}/n\mathbb{Z})^*$, which is abelian (hence solvable).

Example: $\Phi_5(x) = x^4 + x^3 + x^2 + x + 1$ has degree 4, Galois group $\cong \mathbb{Z}/4\mathbb{Z}$, solvable by radicals.

Example

Example 5: Pure Polynomials $x^n - a$

Polynomials of the form $x^n - a$ are always solvable by radicals.

Reason: Roots are $\sqrt[n]{a} \cdot \zeta_n^k$ for $k = 0, 1, \ldots, n-1$ where ζ_n is a primitive nth root of unity. Splitting field is $\mathbb{Q}(\sqrt[n]{a}, \zeta_n)$, obtained by radical extensions.

Example: $x^5 - 2$ has Galois group contained in $\mathbb{Z}/5\mathbb{Z} \rtimes (\mathbb{Z}/5\mathbb{Z})^*$, which is solvable.

Example

Example 6: Polynomials with Dihedral Galois Groups

If $\mathrm{Gal}(f) \cong D_n$ (dihedral group), then f is solvable by radicals.

Reason: All dihedral groups are solvable.

Example: Many quintics have Galois group D_5 (order 10), which is solvable. Such quintics *can* be solved by radicals, even though the general quintic cannot.

Non-Solvable Examples

Example

Example 7: Specific Non-Solvable Quintics

The following quintics are NOT solvable by radicals:

- $x^5 - x - 1$ (Gal = S_5)
- $x^5 - x^2 - 1$ (Gal = S_5)
- $x^5 - 4x + 2$ (Gal = S_5)

Each has Galois group S_5, which is not solvable.

Example

Example 8: Higher Degree Polynomials

For $n \geq 5$, the general degree n polynomial is not solvable by radicals (Galois group S_n).

But specific polynomials may still be solvable if they have special structure (reducibility, abelian Galois groups, etc.).

Computational Approach

Algorithm to determine solvability:

1. Compute the Galois group $G = \text{Gal}(f)$ (algorithmic, but complex)
2. Determine if G is solvable:
   - Compute the derived series $G, G', G'', \ldots$
   - Check if it reaches $\{e\}$
3. If G is solvable, then f is solvable by radicals; otherwise not

Computer algebra systems (Magma, GAP, SageMath) can perform these computations for many polynomials.

Greek Meditation: Necessity and Impossibility

The ancient Greeks distinguished between what is necessary and what is contingent, between what must be and what merely happens to be. In the realm of mathematics, we find a purer form of necessity—the necessity of logical consequence.

The impossibility of solving the general quintic by radicals is not a limitation of human ingenuity, but a necessity written into the structure of mathematics itself. It is as impossible as constructing a square circle, as necessary as the fact that two plus two equals four.

Aristotle taught us about potentiality and actuality. The quintic has roots—they exist, they are actual. We can approximate them numerically to arbitrary precision. But the potential to express them using radicals does not exist. This is not a failure but a revelation: mathematics has boundaries, and these boundaries are themselves objects of mathematical study.

Consider the great impossibility proofs of Greek geometry: the impossibility of squaring the circle, trisecting an arbitrary angle, or doubling the cube using only compass and straightedge. These were not proved in antiquity, but they were conjectured. They reveal the same truth: mathematics is not a realm where all that can be imagined can be achieved. There are necessary limitations.

In recognizing impossibility, we achieve a higher understanding. We see not just what is, but what must be and what cannot be. This is the wisdom of limits, the philosophy of necessity. When we prove that something cannot be done, we have learned something profound about the structure of reality—or at least, the structure of mathematical reality.

The unsolvable quintic stands as a monument to this wisdom. It reminds us that mathematics is not merely a tool for calculation, but a window into necessary truth. And in contemplating what cannot be, we come to understand more deeply what is.

Zen Meditation: The Unsolvable Quintic

The student asks the master: "How do I solve the quintic?"

The master replies: "By not solving it."

The student is confused: "But the roots exist. I can approximate them numerically. I can plot the function and see where it crosses the axis. Why can I not express them exactly using radicals?"

The master smiles: "You seek to grasp water with your fist. The tighter you squeeze, the more it escapes. The quintic's roots exist, yet they cannot be captured in the net

of radicals. This is not a deficiency but a teaching."

"What teaching?" asks the student.

"The teaching of not-knowing. The wisdom of accepting what cannot be grasped. In mathematics, as in life, there are truths that exist beyond our formulas, realities that transcend our symbols. The unsolvable quintic is a koan: it points to the limits of symbolic manipulation while affirming the existence of mathematical truth beyond those limits."

The student ponders: "So the quintic has roots, but they cannot be written in the language of radicals?"

"Just so," says the master. "The roots are like the sound of one hand clapping—they are real, but they cannot be expressed in the expected form. When you seek a radical formula for the general quintic, you are seeking the sound of one hand. The impossibility is not in the mathematics, but in the seeking itself."

"But then how do I work with these roots?"

"You work with them by letting go of the demand for radical expressions. You use numerical methods. You study the Galois group. You understand the structure without insisting on a formula. This is mathematical enlightenment: accepting the thing as it is, not as you wish it to be."

The student asks: "Is this not defeat?"

The master laughs: "Only if you see mathematics as conquest. The quintic has not defeated you. It has taught you. Before you studied Galois theory, you thought all polynomials must be solvable by radicals. Now you know better. Your understanding has deepened. You have moved from ignorance through confusion to wisdom. This is the path."

"When you stop trying to solve the unsolvable, you have solved it. When you accept the impossibility, you have achieved understanding. This is the gate of no-gate, the solution of no-solution."

The student bows, enlightened. In accepting what cannot be grasped, the student has grasped everything.

Practice Problems

Problems on Radical Extensions

Problem 1. Show that $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$ is a radical extension by constructing an explicit radical tower.

Problem 2. Prove that $\mathbb{Q}(\sqrt[3]{2}, \omega)/\mathbb{Q}$ is a radical extension, where $\omega = e^{2\pi i/3}$ is a primitive cube root of unity.

Problem 3. Show that the composition of two radical extensions is a radical extension.

Problem 4. Give an example of a radical extension that is not a Galois extension.

Problem 5. Prove that if K/F is a radical extension and L is an intermediate field ($F \subseteq L \subseteq K$), then K/L is a radical extension.

Problems on Solvable Groups

Problem 6. Prove that every subgroup of a solvable group is solvable.

Problem 7. Prove that every quotient of a solvable group is solvable.

Problem 8. Show that D_4 (the dihedral group of order 8) is solvable by exhibiting a composition series with abelian quotients.

Problem 9. Compute the derived series for S_3 and verify it reaches the identity.

Problem 10. Prove that if $N \triangleleft G$ with both N and G/N solvable, then G is solvable.

Problems on Galois Groups and Solvability

Problem 11. Show that the polynomial $x^4 - 2$ is solvable by radicals by:

a. Computing its Galois group over $\mathbb{Q}$

b. Showing the Galois group is solvable

c. Constructing an explicit radical expression for the roots

Problem 12. Determine whether $x^5 - 2$ is solvable by radicals.

Problem 13. Explain why the cyclotomic polynomial $\Phi_{12}(x)$ is solvable by radicals, even though it has degree $\phi(12) = 4$.

Problem 14. Consider $f(x) = x^5 - 5x + 12 \in \mathbb{Q}[x]$. If its Galois group is S_5, prove that f is not solvable by radicals.

Problem 15. Show that any polynomial with Galois group D_5 (dihedral group of order 10) is solvable by radicals.

Problems on the Quintic

Problem 16. Verify that $f(x) = x^5 - x - 1$ has exactly one real root by analyzing the function.

Problem 17. Explain why having exactly 3 real roots and 2 complex roots helps show that a quintic has Galois group containing a transposition.

Problem 18. Prove that the polynomial $x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1$ (which factors as $(x^2 - x - 1)(x^3 + 2x^2 - x - 1)$) is solvable by radicals.

Theoretical Problems

Problem 19. Prove that if $\text{char}(F) = 0$ and K/F is a finite extension contained in some radical extension of F, then K is contained in a Galois radical extension of F.

Problem 20. (Challenge) Prove that for $n \geq 5$, there exist infinitely many polynomials of degree n over $\mathbb{Q}$ that are not solvable by radicals.

* * *

Solutions to Selected Problems

Solution to Problem 1

Problem: Show that $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$ is a radical extension.

Solution: We construct a radical tower:

$$\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt{2}, \sqrt{3})$$

Verify each step is radical:

- **Step 1:** $\mathbb{Q}(\sqrt{2}) = \mathbb{Q}(\alpha_1)$ where $\alpha_1 = \sqrt{2}$ satisfies $\alpha_1^2 = 2 \in \mathbb{Q}$ ✓
- **Step 2:** $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2})(\alpha_2)$ where $\alpha_2 = \sqrt{3}$ satisfies $\alpha_2^2 = 3 \in \mathbb{Q}(\sqrt{2})$ ✓

Both steps adjoin a square root of an element from the previous field, so this is a radical tower. Therefore $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$ is a radical extension. ∎

Solution to Problem 2

Problem: Prove that $\mathbb{Q}(\sqrt[3]{2}, \omega)/\mathbb{Q}$ is a radical extension, where $\omega = e^{2\pi i/3}$.

Solution: Construct the tower:

$$\mathbb{Q} \subseteq \mathbb{Q}(\sqrt[3]{2}) \subseteq \mathbb{Q}(\sqrt[3]{2}, \omega)$$

Verify each step:

- **Step 1:** $\mathbb{Q}(\sqrt[3]{2}) = \mathbb{Q}(\alpha_1)$ where $\alpha_1^3 = 2 \in \mathbb{Q}$ ✓
- **Step 2:** ω satisfies $\omega^3 = 1$. But we need $\omega^n \in \mathbb{Q}(\sqrt[3]{2})$ for some n. Since $\omega^3 = 1 \in \mathbb{Q} \subseteq \mathbb{Q}(\sqrt[3]{2})$, we have $\omega^3 \in \mathbb{Q}(\sqrt[3]{2})$ ✓

Alternatively, we could first adjoin ω: $\mathbb{Q} \subseteq \mathbb{Q}(\omega) \subseteq \mathbb{Q}(\omega, \sqrt[3]{2})$. Either way works. ∎

Solution to Problem 6

Problem: Prove that every subgroup of a solvable group is solvable.

Solution: Let G be solvable with composition series:

$$\{e\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$$

where each G_{i+1}/G_i is abelian.

Let $H \leq G$ be a subgroup. Consider the series:

$$\{e\} = H \cap G_0 \subseteq H \cap G_1 \subseteq \cdots \subseteq H \cap G_n = H$$

By the second isomorphism theorem:

$$(H \cap G_{i+1})/(H \cap G_i) \cong (H \cap G_{i+1})G_i/G_i \subseteq G_{i+1}/G_i$$

Since G_{i+1}/G_i is abelian, so is any subgroup of it. Therefore each quotient $(H \cap G_{i+1})/(H \cap G_i)$ is abelian.

We can refine this to a composition series for H with all abelian quotients, so H is solvable. ∎

Solution to Problem 8

Problem: Show that D_4 is solvable.

Solution: Recall $D_4 = \langle r, s : r^4 = s^2 = e, srs = r^{-1} \rangle$ has order 8.

Let $R = \langle r \rangle$ be the subgroup of rotations (cyclic, order 4). Consider:

$$\{e\} \triangleleft \langle r^2 \rangle \triangleleft R \triangleleft D_4$$

Check normality and quotients:

- $\langle r^2 \rangle = \{e, r^2\}$ is normal in R (subgroup of abelian group)
- R is normal in D_4 (index 2 subgroup)
- $\langle r^2 \rangle$ is normal in D_4 (can verify: $srs = r^{-2} = r^2$)

Quotients:

- $\langle r^2 \rangle / \{e\} \cong \mathbb{Z}/2\mathbb{Z}$ (abelian) ✓
- $R/\langle r^2 \rangle \cong \mathbb{Z}/2\mathbb{Z}$ (abelian) ✓
- $D_4/R \cong \mathbb{Z}/2\mathbb{Z}$ (abelian) ✓

All quotients are abelian, so D_4 is solvable. ∎

Solution to Problem 9

Problem: Compute the derived series for S_3.

Solution: Start with $S_3 = \{e, (12), (13), (23), (123), (132)\}$.

First derived subgroup S_3': The commutator subgroup consists of all commutators $[a, b] = aba^{-1}b^{-1}$.

Compute some commutators:

- $[(12), (13)] = (12)(13)(12)(13) = (123)$
- $[(12), (23)] = (12)(23)(12)(23) = (132)$

These generate $A_3 = \{e, (123), (132)\}$. One can verify $S_3' = A_3$.

Second derived subgroup $(S_3')' = A_3'$: Since $A_3 \cong \mathbb{Z}/3\mathbb{Z}$ is abelian, all commutators are trivial: $A_3' = \{e\}$.

Derived series:

$$S_3 \supseteq A_3 \supseteq \{e\}$$

The series reaches the identity after 2 steps, confirming S_3 is solvable. ∎

Solution to Problem 11

Problem: Show that $f(x) = x^4 - 2$ is solvable by radicals.

Solution:

(a) Galois group: The roots are $\pm\sqrt[4]{2}, \pm i\sqrt[4]{2}$. The splitting field is:

$$K = \mathbb{Q}(\sqrt[4]{2}, i)$$

We have $[K : \mathbb{Q}] = 8$ (since $[\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}] = 4$ and $[K : \mathbb{Q}(\sqrt[4]{2})] = 2$).

The Galois group is $\mathrm{Gal}(K/\mathbb{Q}) \cong D_4$ (dihedral group of order 8).

(b) Solvability of D_4: We showed in Problem 8 that D_4 is solvable.

(c) Explicit radical expressions: The roots are:

$$\sqrt[4]{2}, \quad -\sqrt[4]{2}, \quad i\sqrt[4]{2}, \quad -i\sqrt[4]{2}$$

These are expressed using radicals: the fourth root $\sqrt[4]{2}$ and the imaginary unit $i = \sqrt{-1}$.

More explicitly, starting from $\mathbb{Q}$:

- Adjoin $\sqrt{2}$: first radical
- Adjoin $\sqrt[4]{2} = \sqrt{\sqrt{2}}$: second radical (nested)
- Adjoin $i = \sqrt{-1}$: third radical

Therefore f is solvable by radicals. ∎

Solution to Problem 12

Problem: Determine whether $x^5 - 2$ is solvable by radicals.

Solution: The roots are $\sqrt[5]{2} \cdot \zeta_5^k$ for $k = 0, 1, 2, 3, 4$, where $\zeta_5 = e^{2\pi i/5}$ is a primitive 5th root of unity.

The splitting field is:

$$K = \mathbb{Q}(\sqrt[5]{2}, \zeta_5)$$

Build this as a tower:

$$\mathbb{Q} \subseteq \mathbb{Q}(\zeta_5) \subseteq \mathbb{Q}(\zeta_5, \sqrt[5]{2})$$

The first extension $\mathbb{Q}(\zeta_5)/\mathbb{Q}$ is cyclotomic with Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q}) \cong (\mathbb{Z}/5\mathbb{Z})^* \cong \mathbb{Z}/4\mathbb{Z}$ (abelian).

The second extension $\mathbb{Q}(\zeta_5, \sqrt[5]{2})/\mathbb{Q}(\zeta_5)$ adjoins a 5th root, and $\mathbb{Q}(\zeta_5)$ contains 5th roots of unity. By Kummer theory, this extension is cyclic (abelian).

Therefore $\mathrm{Gal}(K/\mathbb{Q})$ has a composition series with abelian quotients, so it's solvable.

Conclusion: $x^5 - 2$ IS solvable by radicals. ∎

Note: This is very different from $x^5 - x - 1$! The polynomial $x^5 - 2$ has special structure (pure polynomial) giving it an abelian/solvable Galois group, while $x^5 - x - 1$ has Galois group S_5.

Solution to Problem 15

Problem: Show that any polynomial with Galois group D_5 is solvable by radicals.

Solution: The dihedral group D_5 has order 10: $D_5 = \langle r, s : r^5 = s^2 = e, srs = r^{-1} \rangle$.

Let $R = \langle r \rangle$ (cyclic subgroup of order 5). Then $R \triangleleft D_5$ (index 2, hence normal).

Composition series:

$$\{e\} \triangleleft R \triangleleft D_5$$

Quotients:

- $R/\{e\} \cong \mathbb{Z}/5\mathbb{Z}$ (abelian) ✓
- $D_5/R \cong \mathbb{Z}/2\mathbb{Z}$ (abelian) ✓

Both quotients are abelian, so D_5 is solvable.

By Galois' theorem, any polynomial with Galois group D_5 is solvable by radicals. ∎

Solution to Problem 16

Problem: Verify that $f(x) = x^5 - x - 1$ has exactly one real root.

Solution: Use calculus and the intermediate value theorem.

Step 1: Find critical points.

$$f'(x) = 5x^4 - 1 = 0 \implies x^4 = \frac{1}{5} \implies x = \pm\sqrt[4]{\frac{1}{5}}$$

So f has two critical points: $x = \pm 5^{-1/4} \approx \pm 0.669$.

Step 2: Evaluate f at critical points and test points.

- $f(-2) = -32 + 2 - 1 = -31 < 0$
- $f(-5^{-1/4}) \approx (-0.669)^5 - (-0.669) - 1 \approx -0.13 + 0.67 - 1 = -0.46 < 0$
- $f(0) = -1 < 0$
- $f(5^{-1/4}) \approx (0.669)^5 - 0.669 - 1 \approx 0.13 - 0.67 - 1 = -1.54 < 0$
- $f(1) = 1 - 1 - 1 = -1 < 0$
- $f(2) = 32 - 2 - 1 = 29 > 0$

Step 3: Count sign changes. The function is negative at both critical points and at all points left of 2, but positive at $x = 2$. Since f is continuous, there is exactly one root between $x = 1$ and $x = 2$.

Step 4: Behavior at infinity. As $x \to -\infty$, $f(x) \to -\infty$. As $x \to +\infty$, $f(x) \to +\infty$. Combined with the critical point analysis, there is exactly one crossing of the x-axis.

Conclusion: f has exactly one real root (between 1 and 2). Since degree is 5 (odd), the other 4 roots are complex: two pairs of complex conjugates. ∎

Solution to Problem 18

Problem: Prove that $f(x) = x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1$ is solvable by radicals, given that it factors as $(x^2 - x - 1)(x^3 + 2x^2 - x - 1)$.

Solution: Since f factors over $\mathbb{Q}$, its splitting field is the compositum of the splitting fields of the two factors.

Factor 1: $x^2 - x - 1$ has degree 2. All quadratics are solvable by radicals (quadratic formula). Roots:

$$x = \frac{1 \pm \sqrt{5}}{2}$$

Factor 2: $x^3 + 2x^2 - x - 1$ has degree 3. All cubics are solvable by radicals (Cardano's formula), since S_3 is solvable.

Compositum: The splitting field of f is obtained by composing two solvable extensions. Since the compositum of solvable extensions (under appropriate conditions) is solvable, f is solvable by radicals.

More directly: we can write all five roots using radicals (quadratic formula for the first two, cubic formula for the other three), so f is explicitly solvable by radicals. ∎

Solution to Problem 19

Problem: Prove that if $\text{char}(F) = 0$ and K/F is contained in some radical extension of F, then K is contained in a Galois radical extension of F.

Solution sketch: Let $K \subseteq L$ where L/F is radical. In characteristic 0, we can assume L/F is separable (by taking separable closures).

Let M be the normal closure of L over F (adjoin all conjugates of elements of L). Then M/F is Galois.

Claim: M/F is radical.

If L has radical tower $F = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = L$ with $K_{i+1} = K_i(\alpha_i)$ and $\alpha_i^{m_i} \in K_i$, then M is generated over F by all conjugates of all the α_i.

Let M_i be the normal closure of K_i over F. Then:

- M_{i+1} is generated by M_i and all conjugates of α_i
- Each conjugate $\sigma(\alpha_i)$ satisfies $\sigma(\alpha_i)^{m_i} = \sigma(\alpha_i^{m_i}) \in M_i$
- So M_{i+1}/M_i is obtained by adjoining finitely many m_i-th roots

This can be refined to a radical tower. Therefore M/F is a Galois radical extension containing K. ∎

Solution to Problem 20

Problem: Prove that for $n \geq 5$, there exist infinitely many polynomials of degree n over $\mathbb{Q}$ that are not solvable by radicals.

Solution: It suffices to show there are infinitely many irreducible polynomials of degree n over $\mathbb{Q}$ with Galois group S_n (since S_n is not solvable for $n \geq 5$).

Construction: For each prime p, consider:

$$f_p(x) = x^n - p$$

Wait, these have abelian Galois groups. Let's use a different construction.

Better approach: Consider the family of polynomials:

$$g_t(x) = x^n - x - t$$

for varying $t \in \mathbb{Q}$.

For "most" values of t, one can show g_t has Galois group S_n. More precisely:

Theorem (Hilbert Irreducibility): For infinitely many values of $t \in \mathbb{Z}$, the polynomial $g_t(x) = x^n - x - t$ is irreducible over $\mathbb{Q}$ with Galois group S_n.

This is a consequence of Hilbert's irreducibility theorem, which guarantees that specializations of generic polynomials preserve "generic" properties (like having full symmetric group as Galois group) for infinitely many specializations.

Since S_n is not solvable for $n \geq 5$, each such polynomial is not solvable by radicals. Therefore there are infinitely many non-solvable degree n polynomials over $\mathbb{Q}$. ∎

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Thus concludes our study of solvability by radicals—
the bridge between fields and groups,
the explanation of impossibility,
and Galois' enduring gift to mathematics.

Chapter 19: Classical Problems in Galois Theory

"For two thousand years they sought, with compass and straightedge alone,
until Galois theory revealed: some ideals lie beyond our tools."

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19.1 The Three Classical Greek Problems

In the golden age of Greek mathematics (roughly 500–300 BCE), geometers investigated constructions using only two tools: an unmarked straightedge and a compass. With these simple instruments, they achieved remarkable feats: bisecting angles, constructing regular polygons, finding geometric means.

But three problems resisted all attempts, becoming legendary challenges that would occupy mathematicians for over two millennia:

1. Doubling the Cube (The Delian Problem): Given a cube of volume V, construct a cube of volume $2V$ using compass and straightedge.

2. Trisecting the Angle: Given an arbitrary angle θ, construct an angle $\theta/3$ using compass and straightedge.

3. Squaring the Circle: Given a circle of radius r, construct a square with the same area using compass and straightedge.

Legend says the Delian problem arose when the oracle at Delos commanded that Apollo's cubic altar be doubled in size to appease the gods. Greek mathematicians tried every method they knew—all failed.

For over 2000 years, these problems remained unsolved. Many believed solutions would eventually be found. But in the 19th and early 20th centuries, mathematicians proved the shocking truth: *all three problems are impossible*.

The key insight came from Galois theory, which transforms geometric questions into algebraic ones. A construction is possible with compass and straightedge if and only if the corresponding number lies in a field extension of degree 2^k over $\mathbb{Q}$. This simple criterion immediately rules out all three classical problems.

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19.2 Constructible Numbers

What numbers can be constructed with compass and straightedge? We start with the integers (marked on a line), and each construction either adds, subtracts, multiplies, divides, or extracts square roots.

Definition

Definition 19.1 (Constructible Number)
A real number α is **constructible** if, starting from points at 0 and 1 on a line, we can construct a line segment of length $|\alpha|$ using only compass and straightedge in finitely many steps.

Basic constructions allowed:

- Draw a line through two existing points
- Draw a circle with center at an existing point, passing through another existing point
- Mark intersections of lines and circles

Theorem

Theorem 19.2 (Algebraic Characterization of Constructibility)
A real number α is constructible if and only if α lies in a field extension of $\mathbb{Q}$ that can be built as a tower of quadratic extensions:

$$\mathbb{Q} = F_0 \subset F_1 \subset \cdots \subset F_n$$

where $[F_{i+1} : F_i] = 2$ for each i, and $\alpha \in F_n$.

Proof.
Proof (Sketch).

($\Longrightarrow$): Each construction step (intersection of lines/circles) involves solving at most quadratic equations. Coordinates of new points lie in quadratic extensions of previously constructed points' coordinates. Thus constructible numbers lie in towers of quadratic extensions.

($\Longleftarrow$): Conversely, if $\alpha \in F_n$ where each $F_{i+1} = F_i(\sqrt{\beta_i})$, we can geometrically construct $\sqrt{\beta_i}$ from β_i (standard compass-straightedge construction). Building up the tower gives α. □

∎

Corollary 19.3
If α is constructible, then $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 2^k$ for some $k \geq 0$.

Proof.
Proof.
By the tower law, $[F_n : \mathbb{Q}] = [F_n : F_{n-1}] \cdots [F_1 : F_0] = 2^n$. Since $\alpha \in F_n$, we have $\mathbb{Q}(\alpha) \subseteq F_n$, so $[\mathbb{Q}(\alpha) : \mathbb{Q}]$ divides 2^n, hence is a power of 2. □ ∎

This gives us the key criterion: **if the degree $[\mathbb{Q}(\alpha) : \mathbb{Q}]$ is not a power of 2, then α is not constructible.**

Example

Example 19.1 (Constructible Numbers)

- All rationals are constructible (use similar triangles)
- $\sqrt{2}$ is constructible: $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2 = 2^1$
- $\sqrt{2} + \sqrt{3}$ is constructible: $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = 4 = 2^2$
- $\sqrt{2 + \sqrt{3}}$ is constructible: lies in $\mathbb{Q}(\sqrt{3}, \sqrt{2 + \sqrt{3}})$ with degree $4 = 2^2$

Example

Example 19.2 (Non-Constructible Numbers)

- $\sqrt[3]{2}$ is **not** constructible: $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$, not a power of 2
- $\cos(\pi/9) = \cos(20^\circ)$ is **not** constructible (we'll prove this below)

19.3 Doubling the Cube

Problem: Given a cube of side length 1 (volume 1), construct a cube of volume 2.

Reduction to algebra: If the new cube has side length x, then $x^3 = 2$, so $x = \sqrt[3]{2}$.

Question: Is $\sqrt[3]{2}$ constructible?

Theorem

Theorem 19.4 (Impossibility of Doubling the Cube)
The number $\sqrt[3]{2}$ is not constructible. Therefore, doubling the cube is impossible with compass and straightedge.

Proof.
Proof.

Step 1: Find the minimal polynomial of $\sqrt[3]{2}$ over $\mathbb{Q}$. Clearly $\sqrt[3]{2}$ is a root of $f(x) = x^3 - 2$.

Step 2: Show $f(x)$ is irreducible over $\mathbb{Q}$. By Eisenstein's criterion with $p = 2$: 2 divides the constant term -2, 2 does not divide the leading coefficient 1, and $2^2 = 4$ does not divide -2. Therefore $f(x)$ is irreducible.

Step 3: Compute the degree. Since f is irreducible of degree 3, we have

$$[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = \deg(f) = 3$$

Step 4: Apply the constructibility criterion. Since $3 = 2^k$ has no integer solution k, the number $\sqrt[3]{2}$ is not constructible.

Conclusion: Doubling the cube is impossible. □

∎

Historical note: The Greeks found many approximate constructions and mechanical solutions (using tools beyond compass and straightedge). But the exact construction they sought—using only their two permitted tools—is provably impossible.

This proof is remarkably simple! The problem that baffled mathematicians for 2300 years yields immediately to Galois theory. The key: translating geometry into algebra, then using field theory to compute degrees.

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19.4 Trisecting the Angle

Problem: Given an arbitrary angle θ, construct an angle $\theta/3$.

Important: Some angles *can* be trisected! For example, 90° trisects to 30° (constructible). The question is whether *every* angle can be trisected.

Strategy: Find one specific angle that cannot be trisected. We'll use $\theta = 60^\circ$.

Theorem

Theorem 19.5 (Impossibility of Angle Trisection)
A 60° angle cannot be trisected using compass and straightedge. Therefore, angle trisection is impossible in general.

Proof.

Proof.

Setup: To trisect 60°, we must construct 20°. Since angles can be constructed if and only if their cosines can be constructed, we must determine if $\alpha = \cos(20^\circ)$ is constructible.

Step 1: Find a polynomial satisfied by α. We use the triple-angle formula:

$$\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$$

Setting $\theta = 20^\circ$, we get $\cos(60^\circ) = 4\cos^3(20^\circ) - 3\cos(20^\circ)$. Since $\cos(60^\circ) = 1/2$:

$$\frac{1}{2} = 4\alpha^3 - 3\alpha$$

Rearranging:

$$8\alpha^3 - 6\alpha - 1 = 0$$

Step 2: Show this polynomial is irreducible. Let $f(x) = 8x^3 - 6x - 1 \in \mathbb{Q}[x]$.

Check for rational roots: By the Rational Root Theorem, possible rational roots are $\pm 1, \pm 1/2, \pm 1/4, \pm 1/8$. Direct calculation:

- $f(1) = 8 - 6 - 1 = 1 \neq 0$
- $f(-1) = -8 + 6 - 1 = -3 \neq 0$
- $f(1/2) = 8(1/8) - 6(1/2) - 1 = 1 - 3 - 1 = -3 \neq 0$
- $f(-1/2) = -1 + 3 - 1 = 1 \neq 0$
- (similar calculations for $\pm 1/4, \pm 1/8$)

None are roots. Since f is cubic with no rational roots, it's irreducible over $\mathbb{Q}$.

Step 3: Compute the degree. Since f is the minimal polynomial of $\alpha = \cos(20^\circ)$:

$$[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$$

Step 4: Apply constructibility criterion. Since 3 is not a power of 2, α is not constructible.

Conclusion: We cannot construct $\cos(20^\circ)$, so we cannot construct a 20° angle, so we cannot trisect 60°. Therefore angle trisection is impossible in general. □

∎

Which Angles Can Be Trisected?

An angle θ can be trisected if and only if $\cos(\theta/3)$ is constructible. This depends on the minimal polynomial of $\cos(\theta/3)$.

Examples of trisectable angles:

- 90° (gives 30°, and $\cos(30^\circ) = \sqrt{3}/2$ is constructible)
- 180° (gives 60°)
- 270° (gives 90°)
- Any angle θ where $\cos(\theta/3)$ has degree 2^k over $\mathbb{Q}$

Examples of non-trisectable angles:

- 60° (as proved above)

- 40° (by similar analysis)
- Most arbitrary angles

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19.5 Squaring the Circle

Problem: Given a circle of radius 1 (area π), construct a square with area π.

Reduction: The square must have side length $\sqrt{\pi}$. So the question becomes: is $\sqrt{\pi}$ constructible?

This is deeper than the previous two problems, because it requires understanding *transcendental* numbers.

> **Definition**
>
> Definition 19.6 (Algebraic and Transcendental)
> A complex number α is **algebraic** if it is the root of some polynomial with rational coefficients. Otherwise, α is **transcendental**.

Examples:

- $\sqrt{2}$ is algebraic (root of $x^2 - 2$)
- $\sqrt[3]{2}$ is algebraic (root of $x^3 - 2$)
- i is algebraic (root of $x^2 + 1$)
- π is transcendental (proved by Lindemann, 1882)
- e is transcendental (proved by Hermite, 1873)

> **Proposition**
>
> Proposition 19.7
> Every constructible number is algebraic.

> ***Proof.***
> Proof.
> If α is constructible, then $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 2^k < \infty$. The minimal polynomial of α over $\mathbb{Q}$ has degree 2^k, so α is algebraic. □ ∎

> **Theorem**
>
> Theorem 19.8 (Lindemann-Weierstrass Theorem)
> The number π is transcendental.

Proof: The proof requires complex analysis and is beyond our scope. Lindemann proved this in 1882, resolving a question that had stood for over 2000 years.

> **Theorem**
>
> Theorem 19.9 (Impossibility of Squaring the Circle)
> Squaring the circle is impossible with compass and straightedge.

> ***Proof.***
> Proof.
>
> **Step 1:** To square the circle, we must construct $\sqrt{\pi}$.
>
> **Step 2:** By Proposition 19.7, every constructible number is algebraic. Therefore, if $\sqrt{\pi}$ were constructible, it would be algebraic.
>
> **Step 3:** If $\sqrt{\pi}$ were algebraic, then $\pi = (\sqrt{\pi})^2$ would also be algebraic (algebraic numbers are closed under field operations).
>
> **Step 4:** But π is transcendental (Theorem 19.8), contradiction.
>
> **Conclusion:** $\sqrt{\pi}$ is not constructible, so squaring the circle is impossible. □

∎

Squaring the circle was the last of the three classical problems to be resolved, requiring deeper mathematics (transcendence theory) than the other two.

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19.6 Constructible Regular Polygons

A beautiful application of Galois theory: determining which regular n-gons can be constructed with compass and straightedge.

Equivalent problem: A regular n-gon is constructible if and only if $\cos(2\pi/n)$ is constructible.

This is related to the nth roots of unity: $\zeta_n = e^{2\pi i/n}$. We have $\cos(2\pi/n) = (\zeta_n + \zeta_n^{-1})/2$.

> **Definition**
>
> Definition 19.10 (Fermat Prime)
> A **Fermat prime** is a prime of the form $F_m = 2^{2^m} + 1$ for some $m \geq 0$.

Known Fermat primes:

- $F_0 = 2^1 + 1 = 3$
- $F_1 = 2^2 + 1 = 5$

- $F_2 = 2^4 + 1 = 17$
- $F_3 = 2^8 + 1 = 257$
- $F_4 = 2^{16} + 1 = 65537$

These are the *only five known* Fermat primes! It's conjectured that no others exist, but this remains unproven.

> **Theorem**
>
> Theorem 19.11 (Gauss-Wantzel Theorem)
> A regular n-gon is constructible with compass and straightedge if and only if
>
> $$n = 2^k p_1 p_2 \cdots p_r$$
>
> where $k \geq 0$ and $p_1, \ldots, p_r$ are distinct Fermat primes.

> ***Proof.***
> Proof (Sketch).
>
> **Key insight:** The n-gon is constructible iff $\zeta_n = e^{2\pi i/n}$ lies in a tower of quadratic extensions of $\mathbb{Q}$.
>
> **Step 1:** The Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^*$ has order $\varphi(n)$ (Euler totient function).
>
> **Step 2:** For constructibility, we need $\varphi(n)$ to be a power of 2 (so the extension can be built by quadratic steps).
>
> **Step 3:** If $n = 2^k p_1^{a_1} \cdots p_r^{a_r}$ with odd primes p_i, then
>
> $$\varphi(n) = 2^{k-1} p_1^{a_1 - 1}(p_1 - 1) \cdots p_r^{a_r - 1}(p_r - 1)$$
>
> For this to be a power of 2, we need:
>
> - Each $a_i = 1$ (no repeated odd prime factors)
> - Each $p_i - 1$ is a power of 2
>
> **Step 4:** If p is odd and $p - 1 = 2^m$, then $p = 2^m + 1$. For p to be prime, m must be a power of 2 (otherwise $2^m + 1$ factors). Thus p is a Fermat prime.
>
> **Conclusion:** Constructibility requires $n = 2^k p_1 \cdots p_r$ with distinct Fermat primes p_i. □

∎

Examples

Constructible n-gons:

- $n = 3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20, 24, 30, 32, 34, 40, 48, 51, 60, 64, 68, 80, 85, 96, \ldots$

Non-constructible n-gons:

- $n = 7, 9, 11, 13, 14, 18, 19, 21, 22, 23, 25, 26, 27, 28, 29, \ldots$

Example

Example 19.3 (Regular Pentagon)

$n = 5 = F_1$ is a Fermat prime, so the regular pentagon is constructible. (The Greeks knew this!)

Example

Example 19.4 (Regular Heptagon)

$n = 7$. We have $\varphi(7) = 6 = 2 \cdot 3$, which is not a power of 2. Therefore the regular heptagon (7-gon) is **not** constructible.

Example

Example 19.5 (Regular 17-gon)

$n = 17 = F_2$ is a Fermat prime. Gauss proved (at age 19, in 1796!) that the regular 17-gon is constructible. This was the first new constructible polygon discovered since antiquity—a triumph that convinced Gauss to pursue mathematics.

Gauss explicitly constructed the 17-gon, though the construction is quite complex (involving many steps). His discovery showed the power of algebraic methods in geometry.

Example

Example 19.6 (Regular 257-gon and 65537-gon)

Since $257 = F_3$ and $65537 = F_4$ are Fermat primes, regular polygons with these numbers of sides are constructible! Of course, the constructions are enormously complex.

In 1894, Johann Gustav Hermes claimed to have constructed the 65537-gon after 10 years of work. His manuscript is preserved in a suitcase at the University of Göttingen.

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Greek Meditation: The Limits of Geometry—When the Compass Falls Short of the Ideal

Plato taught that the physical world is but a shadow of the realm of *Forms* (εἶδος, eidos)—perfect, eternal, unchanging ideals. A circle drawn in sand is an imperfect reflection of the true Circle, which exists in the realm of Forms.

The classical Greek problems—doubling the cube, trisecting the angle, squaring the circle—represent a profound philosophical inquiry: *Can human tools reach the ideal?*

The compass and straightedge were special to the Greeks. Unlike measuring devices (which depend on contingent units), these tools embody pure geometric relationships. The straightedge creates lines (one-dimensional), the compass creates circles (perfect curves). Together, they seemed to capture the essence of geometry itself.

For 2000 years, mathematicians believed that with these ideal tools, any geometric relationship could be constructed. The three classical problems represented not mere challenges, but *tests of the power of pure reason*.

Then, in the 19th century, Galois theory revealed the truth: *Some ideals lie beyond our tools. Some truths cannot be reached, no matter how perfect our methods.*

This is reminiscent of Plato's Allegory of the Cave. The prisoners see only shadows on the wall, never the Forms themselves. Similarly, with compass and straightedge, we can construct $\sqrt{2}$ and $\sqrt{\sqrt{2}}$—quadratic towers leading toward the Forms—but we cannot reach $\sqrt[3]{2}$, which lies in a different direction entirely.

The Greek philosopher Parmenides distinguished between the "Way of Truth" (ἀλήθεια, alētheia) and the "Way of Opinion" (δόξα, doxa). For millennia, it was merely *opinion* that the classical problems were impossible. Only Galois theory elevated this to *truth*—necessary, proven, certain.

Aristotle's concept of *potentiality and actuality* illuminates this. The doubling of the cube exists as a mathematical idea (potential), but it cannot be actualized through compass-and-straightedge construction. The *Form* exists, but our tools cannot reach it.

There is deep wisdom here, applicable beyond mathematics:

- **Tools have inherent limitations.** No matter how perfect the compass, it cannot trisect every angle. Similarly, human reason, however refined, has boundaries.
- **Some truths require different approaches.** We cannot construct $\sqrt[3]{2}$ with compass alone, but we can approximate it numerically, or understand it algebraically. Different tools reveal different aspects of reality.
- **Impossibility is itself a truth.** Knowing what cannot be done is as valuable as knowing what can be done. It redirects effort toward the possible.

The Pythagoreans believed that "all is number"—that geometric relationships could be captured by ratios of integers. The discovery of irrational numbers (√2) shattered this belief, but the faith in geometric constructibility remained. Now Galois theory shatters this too: not all numbers can be reached by geometric means.

Yet there is beauty in limitation. The Gauss-Wantzel theorem doesn't say "we fail to construct most polygons"—it says "these specific polygons (those with Fermat prime factors) are the blessed ones, touched by constructibility." The 17-gon is special *because* most n-gons are not constructible.

Plato would say: The Form of the 17-gon is accessible to our tools; the Form of the 7-gon is not. Both Forms exist eternally, but our mortal instruments can grasp only some.

In the *Republic*, Plato describes the "Divided Line"—levels of reality from shadows to Forms, from opinion to knowledge. Galois theory elevates us on this line. We move from:

- *Belief* (maybe it's impossible) to
- *Knowledge* (provably impossible) to

- *Understanding* (comprehending why: degree not a power of 2)

The highest knowledge, for Plato, is knowledge of the Good—understanding not just *that* something is true, but *why* it must be true. Galois theory provides this: not just "the cube cannot be doubled," but "because $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$, and constructible numbers require degree 2^k."

In acknowledging the limits of our tools, we glimpse the infinite structure of reality itself. The impossibility of the classical problems reveals something true and eternal about the nature of number, geometry, and being.

As Socrates taught: "The only true wisdom is knowing you know nothing." Applied here: The only true mathematical mastery is knowing what cannot be done—and understanding *why*.

Zen Meditation: Impossibility as Liberation—The Gate of No-Gate

The koan of the compass and straightedge: *How do you trisect an angle that cannot be trisected?*

The student struggles, trying method after method, construction after construction. Each fails. The student becomes frustrated. "There must be a way! I simply haven't found it yet."

The master says: "There is no way."

"But master, surely—"

"*There is no way.*"

This is the Zen teaching of *mu* (無)—nothingness, emptiness, the void. When asked "Does a dog have Buddha-nature?", Master Zhaozhou answered simply: "*Mu*." Not yes, not no—*the question dissolves*.

Similarly: "Can we trisect 60°?" The answer is *mu*. The question, posed in terms of compass and straightedge, has no answer—because the answer lies outside the framework of the question.

In Zen, we speak of the "gate of no-gate" (無門關, Wumenguan)—the barrier that is no barrier. What is the gate that has no gate? *Impossibility itself.*

When you realize the angle cannot be trisected, you stop trying to trisect it. This is liberation! You are freed from futile effort. The impossible task, once recognized as impossible, ceases to bind you.

The gate of no-gate opens when you cease trying to pass through.

Master Dōgen wrote of *shikantaza*—"just sitting," meditation without goal or purpose. Applied to mathematics: *just understanding*, without demanding that all problems be solvable, all questions answerable.

The ancient Greek geometers, for 2000 years, pushed against the wall of impossibility. They pushed and pushed, believing breakthrough was imminent. Galois theory showed: *the wall is not wall*. There is nothing to push against. The trisection you seek does not exist.

This is like the Zen teaching that "all phenomena are empty" (色即是空, shiki soku ze kū). The formula you seek is empty—it has no inherent existence. Ceasing to grasp for it, you are free.

In the *Diamond Sutra*, the Buddha teaches: "All composite things are like a dream, a phantasm, a bubble, and a shadow." The imagined trisection-method is like this—a phantom, sought but never found, because it never existed.

Yet there is profound value in the seeking! The 2000-year quest to solve the classical problems generated enormous mathematical progress: new methods, deeper understanding, the birth of algebra and field theory. The journey mattered, even though the destination was a mirage.

This is the paradox: *In striving for the impossible, we achieve the actual.*

Zen Master Linji taught: "If you meet the Buddha on the road, kill him." Meaning: do not cling to any form, even the sacred. Applied here: If you meet the trisection formula, abandon it. Do not cling to the hope of what cannot be. Let go, and find freedom.

The *Heart Sutra* says: "Gate, gate, pāragate, pārasaṃgate, bodhi svāhā"—"Gone, gone, gone beyond, gone utterly beyond, awakening!" Where have we gone? Beyond the constraints of compass and straightedge, into the realm of pure understanding.

We cannot construct $\sqrt[3]{2}$ geometrically. But we can:

- Compute it numerically (≈ 1.26)
- Understand it algebraically (root of $x^3 - 2$)
- Manipulate it symbolically
- Comprehend its structure through Galois theory

Impossibility in one realm opens possibility in another. This is the gate of no-gate.

When Master Huineng heard the phrase "without dwelling on anything, awaken the mind," he was enlightened. Applied to mathematics: *Without dwelling on the impossible construction, awaken to the deeper truth.*

The impossible construction teaches us:

- Non-attachment to method
- Acceptance of limitation
- Appreciation for what *can* be done
- Understanding that different approaches reveal different truths

In a famous koan, a student asks: "What is the way?" The master points. "That's just a finger pointing," objects the student. The master replies: "If you fixate on the finger, you miss the moon."

The compass and straightedge are fingers pointing at geometric truth. If we fixate on these tools alone, we miss the larger mathematical moon—field theory, Galois theory, the structure of the constructible numbers.

The boundary that seems to constrain us is the very teaching that liberates us.

As the *Platform Sutra* teaches: "Fundamentally, not one thing exists." Fundamentally, there is no trisection formula, no cube-doubling construction, no circle-squaring method. And in realizing this fundamental emptiness, we are freed.

The gate of no-gate: *Impossibility itself is the opening to deeper truth.*

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Practice Problems

Problem 19.1. Show that $\sqrt{3+\sqrt{2}}$ is constructible.

> ***Solution.***
> Solution 19.1.
> Build the tower $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\sqrt{2}, \sqrt{3+\sqrt{2}})$. Each step has degree 2, giving total degree $2^2 = 4$, a power of 2. Therefore constructible.

Problem 19.2. Prove that $\sqrt[3]{5}$ is not constructible.

> ***Solution.***
> Solution 19.2.
> Minimal polynomial is $x^3 - 5$, irreducible by Eisenstein. Thus $[\mathbb{Q}(\sqrt[3]{5}) : \mathbb{Q}] = 3$, not a power of 2. Not constructible.

Problem 19.3. Show that the regular 9-gon is not constructible.

> ***Solution.***
> Solution 19.3.
> We have $n = 9 = 3^2$. Since $9 = 2^k p_1 \cdots p_r$ would require 9 to have only Fermat prime factors (and powers of 2), but 3^2 has repeated prime factor 3 (not a Fermat prime in proper form), the 9-gon is not constructible. Alternatively, $\varphi(9) = 6$ is not a power of 2.

Problem 19.4. Verify that the regular 15-gon is constructible.

> ***Solution.***
> Solution 19.4.
> We have $15 = 3 \cdot 5 = F_0 \cdot F_1$, a product of distinct Fermat primes. By Gauss-Wantzel, the regular 15-gon is constructible.

Problem 19.5. Show that $\cos(40^\circ)$ is not constructible.

> ***Solution.***
> Solution 19.5.
> If $\cos(40^\circ)$ were constructible, we could construct 20° (by angle halving), but we proved in Theorem 19.5 that $\cos(20^\circ)$ is not constructible. Actually, we need to be more careful: trisecting 60° to get 20° is impossible. The question asks about $\cos(40^\circ)$ directly. Using similar triple-angle analysis shows $[\mathbb{Q}(\cos(40^\circ)) : \mathbb{Q}] = 3$, not a power of 2.

Problem 19.6. Prove that all constructible numbers are algebraic.

> ***Solution.***
> Solution 19.6.
> If α is constructible, then $\alpha \in F_n$ where $[F_n : \mathbb{Q}] = 2^n$ (tower of quadratics). Thus $[\mathbb{Q}(\alpha) : \mathbb{Q}] \leq 2^n < \infty$, so α is algebraic.

Problem 19.7. Show that $\sqrt{2}+\sqrt{3}+\sqrt{5}$ is constructible.

> ***Solution.***
> Solution 19.7.
> Build tower: $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\sqrt{2},\sqrt{3}) \subset \mathbb{Q}(\sqrt{2},\sqrt{3},\sqrt{5})$. Each step degree 2, total degree $2^3 = 8$, a power of 2. Constructible.

Problem 19.8. Is a regular 36-gon constructible?

> ***Solution.***
> Solution 19.8.
> We have $36 = 4 \cdot 9 = 2^2 \cdot 3^2$. This has a repeated odd prime factor (3^2), which violates the Gauss-Wantzel criterion. Therefore the 36-gon is **not** constructible.

Problem 19.9. Prove that $\sqrt[4]{2}$ is constructible.

> ***Solution.***
> Solution 19.9.
> Tower: $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\sqrt[4]{2})$. Each step degree 2 (minimal polynomials x^2-2 and $x^2-\sqrt{2}$). Total degree $2^2 = 4$, power of 2. Constructible.

Problem 19.10. Show that the angle $\pi/7$ cannot be constructed.

> ***Solution.***
> Solution 19.10.
> The regular 7-gon is not constructible ($\varphi(7) = 6$ not a power of 2), so $\cos(2\pi/7)$ is not constructible, hence $\pi/7$ cannot be constructed.

Problem 19.11. Verify that $F_5 = 2^{32}+1 = 4294967297$ is NOT prime.

> ***Solution.***
> Solution 19.11.
> Euler showed $F_5 = 641 \times 6700417$. Therefore F_5 is not prime, so not a Fermat prime.

Problem 19.12. Can you trisect a 90° angle?

> ***Solution.***
> Solution 19.12.
> **Yes!** Trisecting 90° gives 30°, and $\cos(30^\circ) = \sqrt{3}/2$ is constructible (lies in $\mathbb{Q}(\sqrt{3})$, degree 2). So 30° can be constructed, meaning 90° can be trisected.

Problem 19.13. Show that if α and β are constructible, then so is $\alpha+\beta$.

> ***Solution.***
> Solution 19.13.
> Constructible numbers form a field (closed under $+, -, \times, \div$). Therefore $\alpha+\beta$ is constructible.

Problem 19.14. Prove that e is not constructible.

> ***Solution.***
> Solution 19.14.
> Since e is transcendental (Hermite, 1873) and all constructible numbers are algebraic, e is not constructible.

Problem 19.15. Is the regular 51-gon constructible?

> ***Solution.***
> Solution 19.15.
> We have $51 = 3 \times 17 = F_0 \times F_2$, a product of distinct Fermat primes. By Gauss-Wantzel, the 51-gon **is** constructible.

Problem 19.16. Can you trisect a 45° angle?

> ***Solution.***
> Solution 19.16.
> Trisecting 45° gives 15°. We need to check if $\cos(15^\circ)$ is constructible. Using $\cos(15^\circ) = \cos(45^\circ - 30^\circ) = \cos(45^\circ)\cos(30^\circ) + \sin(45^\circ)\sin(30^\circ) = (\sqrt{2}/2)(\sqrt{3}/2) +$. This lies in $\mathbb{Q}(\sqrt{2}, \sqrt{3})$, degree 4. **Yes, constructible!**

Problem 19.17. Show that the regular 7-gon is not constructible using Galois theory.

> ***Solution.***
> Solution 19.17.
> The 7th roots of unity generate $\mathbb{Q}(\zeta_7)$ with $[\mathbb{Q}(\zeta_7) : \mathbb{Q}] = \varphi(7) = 6$. Since $6 = 2 \cdot 3$ is not a power of 2, the extension cannot be built by quadratic towers. Not constructible.

Problem 19.18. Prove that $\sqrt{2 + \sqrt{2 + \sqrt{2}}}$ is constructible.

> ***Solution.***
> Solution 19.18.
> Iteratively build: $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\sqrt{2 + \sqrt{2}}) \subset \mathbb{Q}(\sqrt{2 + \sqrt{2 + \sqrt{2}}})$. Each step degree ≤ 2, total degree ≤ $2^3 = 8$, power of 2. Constructible.

Problem 19.19. Show that tripling the cube (constructing $\sqrt[3]{3}$) is impossible.

> ***Solution.***
> Solution 19.19.
> Minimal polynomial $x^3 - 3$ is irreducible (Eisenstein). Thus $[\mathbb{Q}(\sqrt[3]{3}) : \mathbb{Q}] = 3$, not a power of 2. Not constructible.

Problem 19.20. How did Gauss prove the 17-gon is constructible?

> ***Solution.***

Solution 19.20.

Gauss showed $\zeta_{17} = e^{2\pi i/17}$ lies in a tower of quadratic extensions of $\mathbb{Q}$. Since $\varphi(17) = 16 = 2^4$, and $\mathrm{Gal}(\mathbb{Q}(\zeta_{17})/\mathbb{Q}) \cong (\mathbb{Z}/17\mathbb{Z})^* \cong \mathbb{Z}/16\mathbb{Z}$ is a 2-group, it has a composition series with cyclic groups of order 2. This gives a tower $\mathbb{Q} = F_0 \subset F_1 \subset F_2 \subset F_3 \subset F_4 = \mathbb{Q}(\zeta_{17})$ with $[F_{i+1} : F_i] = 2$. Galois' constructibility criterion (before Galois!) shows ζ_{17} is constructible, hence the 17-gon.

Chapter 20: Applications of Galois Theory

"From polynomial equations to quantum fields,
Galois' vision illuminates the structure of mathematical reality."

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20.1 Cyclotomic Extensions

Among the most beautiful applications of Galois theory are the *cyclotomic extensions*—fields generated by roots of unity. These extensions have been central to number theory for centuries, and Galois theory reveals their elegant structure.

Definition

Definition 20.1 (Roots of Unity and Cyclotomic Polynomials)

Let $n \geq 1$. A **primitive nth root of unity** is a complex number ζ satisfying $\zeta^n = 1$ and $\zeta^k \neq 1$ for $1 \leq k < n$. We write $\zeta_n = e^{2\pi i/n}$.

The **nth cyclotomic polynomial** is

$$\Phi_n(x) = \prod_{\substack{1 \leq k \leq n \\ \gcd(k,n)=1}} (x - \zeta_n^k)$$

the minimal polynomial of ζ_n over $\mathbb{Q}$.

Properties of cyclotomic polynomials:

- $\deg(\Phi_n) = \varphi(n)$, where φ is Euler's totient function

- $\Phi_n(x) \in \mathbb{Z}[x]$ (integer coefficients)
- $\Phi_n(x)$ is irreducible over $\mathbb{Q}$
- $x^n - 1 = \prod_{d|n} \Phi_d(x)$

Example

Example 20.1 (Small Cyclotomic Polynomials)

- $\Phi_1(x) = x - 1$
- $\Phi_2(x) = x + 1$
- $\Phi_3(x) = x^2 + x + 1$
- $\Phi_4(x) = x^2 + 1$
- $\Phi_5(x) = x^4 + x^3 + x^2 + x + 1$
- $\Phi_6(x) = x^2 - x + 1$
- $\Phi_8(x) = x^4 + 1$
- $\Phi_{12}(x) = x^4 - x^2 + 1$

Theorem

Theorem 20.2 (Galois Group of Cyclotomic Extensions)
The Galois group of $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^*$, the multiplicative group of units modulo n:

$$\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^*$$

Proof.

Proof (Sketch).

An automorphism $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ must send ζ_n to another primitive nth root of unity. The primitive nth roots are ζ_n^k where $\gcd(k,n) = 1$.

Define $\sigma_k(\zeta_n) = \zeta_n^k$ for $k \in (\mathbb{Z}/n\mathbb{Z})^*$. This gives a group homomorphism

$$(\mathbb{Z}/n\mathbb{Z})^* \to \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}), \quad k \mapsto \sigma_k$$

One verifies this is an isomorphism. □

∎

Example

Example 20.2 ($\mathrm{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$)

We have $(\mathbb{Z}/5\mathbb{Z})^* = \{1,2,3,4\} \cong \mathbb{Z}/4\mathbb{Z}$ (cyclic of order 4, generated by 2).

The automorphisms are:

- $\sigma_1 : \zeta_5 \mapsto \zeta_5$ (identity)
- $\sigma_2 : \zeta_5 \mapsto \zeta_5^2$
- $\sigma_3 : \zeta_5 \mapsto \zeta_5^3$
- $\sigma_4 : \zeta_5 \mapsto \zeta_5^4$

Therefore $\mathrm{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$, a cyclic group.

Applications to Number Theory

Cyclotomic fields are fundamental in algebraic number theory. They appear in:

- **Fermat's Last Theorem:** Kummer's approach used unique factorization in cyclotomic integers

- **Class field theory:** Cyclotomic extensions are the building blocks of abelian extensions of $\mathbb{Q}$
- **Primality testing:** Properties of Φ_n relate to primality of n
- **Gauss sums:** Evaluate sums involving roots of unity

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20.2 Kummer Extensions

Kummer theory studies extensions obtained by adjoining nth roots, generalizing our earlier observations about radical extensions.

> **Definition**
>
> Definition 20.3 (Kummer Extension)
> Let F be a field containing a primitive nth root of unity ζ_n, and suppose $\mathrm{char}(F)$ does not divide n. A **Kummer extension** of F is an extension $F(\alpha)$ where $\alpha^n \in F$.

> **Theorem**
>
> Theorem 20.4 (Kummer Theory)
> If F contains ζ_n and $K = F(\alpha)$ where $\alpha^n = a \in F$, then:
>
> 1. $\mathrm{Gal}(K/F)$ is cyclic of order dividing n
> 2. There is a pairing between n-power classes in F^* and cyclic extensions of degree dividing n

Kummer theory was instrumental in proving special cases of Fermat's Last Theorem and developing class field theory.

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20.3 Algebraic Number Theory

Galois theory is indispensable in modern algebraic number theory, the study of algebraic integers and number fields.

Definition

Definition 20.5 (Algebraic Integer, Number Field)

An **algebraic integer** is a root of a monic polynomial with integer coefficients. A **number field** is a finite extension of $\mathbb{Q}$.

For a number field K, the **ring of integers** $\mathcal{O}_K$ is the set of all algebraic integers in K.

Example

Example 20.3 (Rings of Integers)

- $\mathcal{O}_{\mathbb{Q}} = \mathbb{Z}$
- $\mathcal{O}_{\mathbb{Q}(i)} = \mathbb{Z}[i]$ (Gaussian integers)
- $\mathcal{O}_{\mathbb{Q}(\sqrt{-5})} = \mathbb{Z}[\sqrt{-5}]$
- $\mathcal{O}_{\mathbb{Q}(\sqrt{5})} = \mathbb{Z}[\frac{1+\sqrt{5}}{2}]$ (golden ratio)

Failure of Unique Factorization

Unlike $\mathbb{Z}$, not all rings of integers have unique factorization!

> **Example**
>
> Example 20.4 (Non-Unique Factorization in $\mathbb{Z}[\sqrt{-5}]$)
>
> In $\mathbb{Z}[\sqrt{-5}]$, we have two different factorizations of 6:
>
> $$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$$
>
> One can verify that 2, 3, $1 \pm \sqrt{-5}$ are all irreducible in $\mathbb{Z}[\sqrt{-5}]$, yet they give different factorizations!

Ideal class group: The failure of unique factorization is measured by the *ideal class group* $\mathrm{Cl}(\mathcal{O}_K)$. When this group is trivial, unique factorization holds.

Galois Theory in Algebraic Number Theory

Decomposition of primes: Galois theory describes how primes split in field extensions. For a Galois extension $K/\mathbb{Q}$ and prime p, the prime factorization of $p\mathcal{O}_K$ is determined by the decomposition group, a subgroup of $\mathrm{Gal}(K/\mathbb{Q})$.

Class field theory: The ultimate achievement of algebraic number theory (Hilbert, Takagi, Artin), describes all abelian extensions of a number field in terms of its arithmetic. Galois groups play the central role.

20.4 Differential Galois Theory

Galois' ideas extend beyond polynomials to *differential equations*—a field called differential Galois theory or Picard-Vessiot theory.

> **Theorem**
>
> Theorem 20.6 (Liouville's Theorem on Integration)
> An elementary function $f(x)$ has an elementary antiderivative if and only if the differential Galois group of the associated differential equation is solvable.

This explains why some functions have elementary antiderivatives while others don't!

> **Example**
>
> Example 20.5 (Non-Elementary Integrals)
>
> $\int e^{-x^2} dx$**:** The Gaussian integral has no elementary antiderivative. Differential Galois theory proves this: the differential Galois group is non-solvable.
>
> $\int \frac{\sin x}{x} dx$**:** The sine integral Si(x) is also non-elementary, proven via differential Galois theory.
>
> $\int e^x dx = e^x$**:** Elementary! The differential Galois group is trivial (solvable).

Differential Galois theory parallels classical Galois theory: instead of field extensions, we have differential field extensions. Instead of polynomial equations, we have linear differential equations. The same core insight—understanding solutions via symmetry groups—applies.

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20.5 Modern Applications

Coding Theory and Error Correction

Reed-Solomon codes: Used in CDs, DVDs, QR codes, and deep space communication. Based on polynomial evaluation over finite fields.

BCH codes: Bose-Chaudhuri-Hocquenghem codes use cyclotomic cosets and minimal polynomials over $\mathbb{F}_{2^m}$.

The Galois group $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \cong \mathbb{Z}/n\mathbb{Z}$ (generated by the Frobenius automorphism $x \mapsto x^q$) is fundamental to understanding these codes.

Cryptography

AES (Advanced Encryption Standard): Uses the finite field $\mathbb{F}_{2^8} \cong \mathbb{F}_2[x]/(x^8 + x^4 + x^3 + x + 1)$. Operations in this field (including multiplicative inverses) are core to AES's security.

Elliptic curve cryptography: Defined over finite fields $\mathbb{F}_p$ or $\mathbb{F}_{2^m}$. Galois theory of these fields is essential for efficient implementation and security analysis.

Physics and Gauge Theory

Particle physics: The Standard Model uses gauge groups SU(3) × SU(2) × U(1). Understanding representations of these groups (which is representation theory, closely related to Galois theory) is crucial.

Quantum field theory: Symmetry groups and their representations determine particle interactions. Galois' vision—understanding structure via symmetry—is realized at the deepest level of physics.

Computer Algebra Systems

Modern systems like Mathematica, Maple, and Sage use Galois theory for:

- **Polynomial factorization:** Algorithms use field extensions and Galois groups
- **Solving polynomial equations:** Determining solvability and finding symbolic solutions
- **Simplification:** Recognizing algebraic dependencies and simplifying expressions

20.6 The Legacy of Galois

Impact on Mathematics

Galois' work in 1832, written frantically the night before his fatal duel at age 20, revolutionized mathematics. His ideas were so advanced that they weren't fully understood until decades later.

Birth of abstract algebra: Before Galois, algebra was about solving equations. After Galois, algebra became the study of *structures*—groups, rings, fields. This shift from computation to abstraction transformed all of mathematics.

Group theory as independent subject: Galois essentially created group theory. The concept of a group as an abstract structure with certain axioms emerged from his work.

Influence on Modern Mathematics

Category theory: The notion of *Galois connections* (pairs of order-reversing maps between posets that are adjoint) generalizes the correspondence in the Fundamental Theorem.

Algebraic topology: The fundamental group $\pi_1(X)$ and covering space theory are "topological Galois theory"—the fundamental group plays the role of the Galois group.

Grothendieck's Galois theory: In algebraic geometry, Grothendieck generalized Galois theory to the study of fundamental groups of schemes and étale cohomology. This is one of the deepest areas of modern mathematics.

Langlands program: This vast web of conjectures connecting number theory, representation theory, and geometry can be viewed as a vast generalization of class field theory and Galois theory.

From Equations to Understanding Structure

The revolutionary insight: *Mathematics is not about computing specific answers, but about understanding underlying structure.*

Before Galois: "Can we solve $x^5 + 2x + 1 = 0$?"
After Galois: "What is the structure of the symmetry group of this equation?"

This shift—from computational to structural thinking—defines modern mathematics. Every area has followed this pattern:

- Topology: from specific spaces to general topological properties
- Analysis: from specific functions to function spaces and operators
- Geometry: from specific shapes to abstract manifolds and categories

Galois' Vision: Mathematics as Study of Symmetry

Perhaps Galois' deepest contribution: *recognizing that symmetry is the key to understanding mathematical structures*.

This idea now pervades all of mathematics and physics:

- Conservation laws in physics come from symmetries (Noether's theorem)
- Classification theorems in mathematics use symmetry groups
- Geometric structures are understood via their automorphism groups

From a 20-year-old's desperate writing on the eve of his death, an entire worldview emerged: *the universe is structure, and structure is revealed through symmetry*.

Conclusion: The Eternal Beauty of Galois Theory

Évariste Galois died at dawn on May 31, 1832, from wounds received in a duel. His last words to his brother Alfred were reportedly: "Don't cry. I need all my courage to die at twenty."

His mathematical legacy, scribbled hastily by candlelight, would take decades to understand fully. But once understood, it changed mathematics forever.

The correspondence between subfields and subgroups, the criterion for solvability, the impossibility proofs for ancient problems—these are not just technical results. They are windows into the deep structure of mathematical reality.

Galois theory reveals that:

- Algebra and geometry are unified through field extensions
- Group theory and field theory are dual aspects of one structure
- Impossibility is not failure but truth—some boundaries are absolute
- Symmetry is the language of structure
- Abstract thinking illuminates concrete problems

In the 21st century, nearly 200 years after Galois' death, his ideas remain central to pure mathematics, applied mathematics, computer science,

physics, and engineering. The theory continues to grow, branching into new areas, revealing new connections.

This is the triumph of Galois' vision: not solving equations, but understanding the eternal structure beneath them.

As we conclude this volume, we honor Évariste Galois—mathematical revolutionary, political radical, tragic hero. He saw deeper than his contemporaries, thought more abstractly, and left a legacy that will endure as long as mathematics itself.

From polynomial roots to the roots of reality itself, Galois theory stands as one of humanity's greatest intellectual achievements.

> **Greek Meditation: The Eternal Return—Galois' Ideas Echoing Through Mathematics**
>
> The Pythagoreans believed in the *Music of the Spheres* (ἁρμονία τῶν σφαιρῶν)—a perfect cosmic harmony underlying all of reality, expressed through mathematical ratios. They heard echoes of this harmony in musical intervals: the octave (2:1), the perfect fifth (3:2), the perfect fourth (4:3).
>
> Galois theory reveals a similar cosmic harmony, but in the structure of fields and groups. The correspondence between intermediate fields and subgroups is not mere coincidence—it is *mathematical harmony*, as perfect as any musical consonance.
>
> The ancient concept of *logos* (λόγος) meant both "ratio" and "reason"—the rational structure of reality. Heraclitus taught that logos is the principle of order in the universe, the pattern behind apparent chaos.

In Galois theory, we see logos manifested: the ratio $[K : F] = |\mathrm{Gal}(K/F)|$ is not just a number, but a *reason*—it explains why the extension has exactly this many symmetries, why the polynomial has exactly these properties.

Plato's theory of *anamnesis* (ἀνάμνησις)—recollection—taught that learning is remembering eternal truths already known to the soul. When we discover the Fundamental Theorem of Galois Theory, are we discovering something new, or recollecting an eternal truth?

The Neoplatonist Plotinus spoke of *emanation*—how all of reality flows from The One through successive levels. In Galois theory, we see a similar emanation:

- From the base field F (the One)
- Through intermediate fields (emanations)
- To the splitting field K (the manifold reality)

And dually, the Galois group G emanates through its subgroups, mirroring perfectly the field tower.

Aristotle's *entelecheia* (ἐντελέχεια)—the realization of potential—applies beautifully. The splitting field K is the *entelecheia* of the polynomial f: the full actualization of all its roots, the complete realization of its potential.

The Stoic concept of *sympatheia* (συμπάθεια)—cosmic sympathy, the interconnection of all things—finds mathematical expression in Galois theory. Every change in the field structure sympathetically resonates in the group structure. They are not separate entities but aspects of one unified whole.

Consider the Greek ideal of *kosmos* (κόσμος)—order, harmony, beauty. The universe is not chaos but cosmos, ordered structure. Galois theory reveals that algebra too is cosmos: beneath the apparent complexity of polynomial equations lies perfect order, expressed through group structure.

The pre-Socratic philosopher Empedocles taught that reality is shaped by two opposing forces: *Love* (φιλία, philia) which unites, and *Strife* (νεῖκος, neikos) which divides. In Galois theory:

- *Love* is the joining operation: field compositum $K_1 K_2$, group intersection $H_1 \cap H_2$
- *Strife* is the division operation: quotient fields, quotient groups

And from their interplay emerges all structure.

Heraclitus wrote: "The way up and the way down are one and the same" (ὁδὸς ἄνω κάτω μία). In Galois theory, this is literally true: the ascending path through field extensions and the descending path through subgroup chains are one and the same structure, viewed from different perspectives.

The concept of *mimesis* (μίμησις)—imitation—was central to Greek aesthetics. Art imitates nature, mathematics imitates ideal forms. But in Galois theory, we transcend mere imitation: the Galois group doesn't just *represent* the field structure—it *is* the field structure, viewed through the lens of symmetry.

Pythagoras taught: "All is number" (ἀριθμός). Modern mathematics has refined this: *All is structure.* And Galois theory reveals that structure is fundamentally about *relationship*—the correspondence, the mapping, the isomorphism between seemingly different domains.

In the *Timaeus*, Plato describes the Demiurge (creator) imposing mathematical order on primordial chaos. Galois is our mathematical Demiurge: he took the chaotic problem of solving equations and revealed the perfect order beneath—the group structure that had been there, hidden, all along.

The Greek ideal of *theoria* (θεωρία)—contemplation, seeing—was the highest form of knowledge. Not practical know-how, but pure understanding, seeing the truth. Galois theory is pure *theoria*: not "how do I compute this root?" but "what *is* the structure of this equation?"

> *In Galois theory, we hear the Pythagorean Music of the Spheres, manifested as the harmony of fields and groups. We see the Platonic Forms, not as distant ideals, but as living structure in the mathematics before us. We witness the logos—the rational principle—governing the algebra of symmetry.*
>
> Évariste Galois died at twenty, but his ideas echo eternally through mathematics, like the Pythagorean spheres singing their cosmic harmony. Each generation rediscovers his insights, each in their own way, and each time the music sounds again—the eternal return of mathematical truth.

> **Zen Meditation: The Circle Complete—Beginning and End as One**
>
> In Zen, the *ensō* (円相)—the circle—represents enlightenment, emptiness, the universe. A single brushstroke, beginning and end meeting, incomplete yet perfect.
>
> Galois theory is mathematical *ensō*. We begin with a polynomial equation—a question. We journey through fields and groups, towers and correspondences. And where do we end? Back at the beginning—understanding the equation, but now seeing its true nature.
>
> The Fundamental Theorem of Galois Theory is like the circle closing: fields map to subgroups, subgroups map to fields. The way down and the way up form a perfect circle. *Beginning and end are one.*
>
> Master Dōgen taught *shushō ittō* (修証一等)—practice and enlightenment are one. In mathematics: *problem and solution are one*. The structure of the equation *is* its solution. The Galois group doesn't just help us solve—it *is* the solution, the complete understanding.

The *Heart Sutra* proclaims: "Form is emptiness, emptiness is form" (色即是空 空即是色, shiki soku ze kū, kū soku ze shiki). Applied to Galois theory:

- *Form* is the field extension—concrete, specific, particular
- *Emptiness* is the abstract group—pure structure, free of content
- They are *one*: form is emptiness, emptiness is form

The koan asks: "What was your original face before your parents were born?" What is the original nature of a polynomial, before any specific field extension? The answer: *the Galois group*—the inherent symmetry structure, existing before any realization in a particular extension.

In Zen archery, the master teaches: "Let the arrow shoot itself." In Galois theory: *let the structure reveal itself*. Don't force solutions—understand symmetry, and solutions arise naturally.

The Zen concept of *ichigo ichie* (一期一会)—"one time, one meeting"—reminds us that each moment is unique, unrepeatable. Yet in mathematics, truths are eternal. How to reconcile? Each *discovery* of Galois theory is unique (Galois in 1832, you today), but the *truth* discovered is timeless. Moment and eternity, united.

Wu wei (無為)—effortless action, going with the flow of nature—is embodied in Galois theory. We don't force polynomials to factor, don't demand that equations be solvable. We understand their nature, accept their structure, work *with* their inherent symmetry.

Master Linji taught four relationships: sometimes take away the subject, sometimes take away the object, sometimes take away both, sometimes take away neither. In Galois theory:

- Take away the field, focus on the group (abstract algebra)
- Take away the group, focus on the field (field theory)
- Take away both—see pure structure (category theory)

- Take away neither—see the correspondence (Galois theory proper)

Each perspective reveals truth.

The *Platform Sutra* teaches: "Originally there is not one thing" (本来無一物, honrai mu ichimotsu). Applied: there is no fundamental distinction between fields and groups, between algebra and geometry, between problem and solution. These are conventional divisions, useful but not ultimate. The *ultimate* is the structure itself, which transcends all categories.

In the ten oxherding pictures (Zen's map of enlightenment), the journey ends with "returning to the marketplace"—back to ordinary life, but transformed by understanding. Similarly, Galois theory takes us on a journey through abstractions—fields, groups, correspondences—and returns us to where we started: polynomial equations. But now we *understand*. The marketplace looks the same, but we see it differently.

The Zen master Yunmen was asked: "What is the way?" He answered: "*Walk on.*" Galois theory is not the end of mathematics—it is a beginning. From here, we walk on: to algebraic geometry, to number theory, to topology, to category theory. Each step forward is a return to the beginning, spiraling deeper into understanding.

Samsara and nirvana are one—a core teaching of Mahayana Buddhism. The cycle of confusion and the state of enlightenment are not two different things. Similarly, the "problem" (confusion) and the "Galois group" (enlightenment) are not different—the group *is* the problem, fully understood.

The Ouroboros—the snake eating its own tail—appears in many traditions. In mathematics, it is the self-referential structure of Galois theory: the field generates the group, the group determines the field. Each creates and is created by the other, in eternal cycle.

Master Hakuin's famous koan: "What is the sound of one hand clapping?" What is the "sound" of a Galois group? *Silence*. Pure structure makes no sound, leaves no trace, yet contains all possibilities. The silent symmetry underlying the noise of computation.

The circle complete: From Évariste Galois, age 20, writing frantically by candlelight, to you, reading these words nearly 200 years later. The same truths, the same structures, the same eternal mathematics. Time passes, the ensō remains. Beginning and end are one.

As you close this volume, remember: *You have not reached the end. You have returned to the beginning, transformed.* The equations you see now, you see with new eyes—eyes that perceive symmetry, structure, the Galois group beneath the surface.

In the stillness of the Galois correspondence, in the perfection of the Fundamental Theorem, in the impossibility proofs and the solvability criteria—there, in mathematics itself, awakening.

The ensō closes. The journey ends. And in that ending, begins.

▮

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Practice Problems

Problem 20.1. Compute $\Phi_7(x)$, the 7th cyclotomic polynomial.

> ***Solution.***
> Solution 20.1.

Using $x^7 - 1 = \Phi_1(x)\Phi_7(x) = (x-1)\Phi_7(x)$, we get $\Phi_7(x) = (x^7-1)/(x-1) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$.

Problem 20.2. Show that $\mathrm{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Solution.

Solution 20.2.

We have $(\mathbb{Z}/8\mathbb{Z})^* = \{1, 3, 5, 7\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (Klein four-group). By Theorem 20.2, $\mathrm{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q}) \cong (\mathbb{Z}/8\mathbb{Z})^*$.

Problem 20.3. Verify that 2 is irreducible but not prime in $\mathbb{Z}[\sqrt{-5}]$.

Solution.

Solution 20.3.

Irreducible: If $2 = \alpha\beta$ with $\alpha, \beta \in \mathbb{Z}[\sqrt{-5}]$, then $N(\alpha)N(\beta) = N(2) = 4$. Possible norms: 1, 2, 4. Check that no element has norm 2, so any factorization must be trivial. **Not prime:** $2 | 6 = (1+\sqrt{-5})(1-\sqrt{-5})$ but $2 \nmid (1+\sqrt{-5})$ and $2 \nmid (1-\sqrt{-5})$.

Problem 20.4. Show that $\mathbb{Z}[i]$ (Gaussian integers) has unique factorization.

Solution.

Solution 20.4.

The ring $\mathbb{Z}[i] = \mathcal{O}_{\mathbb{Q}(i)}$ is a Euclidean domain (Euclidean function: $N(a+bi) = a^2 + b^2$). Every Euclidean domain is a unique factorization domain.

Problem 20.5. Compute $\varphi(12)$ and identify $(\mathbb{Z}/12\mathbb{Z})^*$.

Solution.

Solution 20.5.

$\varphi(12) = \varphi(4)\varphi(3) = 2 \cdot 2 = 4$. Elements: $(\mathbb{Z}/12\mathbb{Z})^* = \{1, 5, 7, 11\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Problem 20.6. Explain why $\int e^{-x^2}\,dx$ has no elementary antiderivative.

Solution.

Solution 20.6.

By Liouville's theorem (differential Galois theory), an elementary function has an elementary antiderivative iff its differential Galois group is solvable. For e^{-x^2}, the differential equation is $y' + 2xy = 0$, whose differential Galois group is non-solvable (related to the error function). Therefore no elementary antiderivative exists.

Problem 20.7. Show that $\mathbb{Z}[\sqrt{-3}]$ is not the ring of integers of $\mathbb{Q}(\sqrt{-3})$.

Solution.

Solution 20.7.

The element $\omega = \frac{-1+\sqrt{-3}}{2}$ satisfies $\omega^2 + \omega + 1 = 0$ (monic polynomial with integer coefficients), so ω is an algebraic integer. But $\omega \notin \mathbb{Z}[\sqrt{-3}]$. Therefore $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})} = \mathbb{Z}[\omega] = \mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ properly contains $\mathbb{Z}[\sqrt{-3}]$.

Problem 20.8. What is $\deg(\Phi_{15}(x))$?

Solution.

Solution 20.8.

$\deg(\Phi_{15}) = \varphi(15) = \varphi(3)\varphi(5) = 2 \cdot 4 = 8$.

Problem 20.9. Show that cyclotomic polynomials have integer coefficients.

Solution.

Solution 20.9.

Proof by induction using $x^n - 1 = \prod_{d|n} \Phi_d(x)$. Base case: $\Phi_1(x) = x - 1 \in \mathbb{Z}[x]$. Inductive step: \(\Phi_n(x) = (x^n - 1) / \prod_{d|n, d

Problem 20.10. Find all primitive 12th roots of unity.

Solution.

Solution 20.10.

They are ζ_{12}^k where $\gcd(k, 12) = 1$, i.e., $k \in \{1, 5, 7, 11\}$. Explicitly: $e^{2\pi i/12}, e^{10\pi i/12}, e^{14\pi i/12}, e^{22\pi i/12}$.

Problem 20.11. Explain how Reed-Solomon codes use Galois theory.

Solution.

Solution 20.11.

RS codes encode data as evaluations of polynomials over finite fields $\mathbb{F}_q$. Decoding uses properties of $\mathbb{F}_q$ and its extensions, including the Galois group structure. Error correction exploits the fact that a polynomial of degree $< k$ is uniquely determined by k evaluations.

Problem 20.12. What is the Frobenius automorphism of $\mathbb{F}_{p^n}/\mathbb{F}_p$?

Solution.

Solution 20.12.

The Frobenius automorphism is $\varphi(x) = x^p$. It generates $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \cong \mathbb{Z}/n\mathbb{Z}$, with $\varphi^n = \mathrm{id}$ (since $x^{p^n} = x$ for all $x \in \mathbb{F}_{p^n}$).

Problem 20.13. Show that $\Phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$ for prime p.

Solution.

Solution 20.13.

Since $x^p - 1 = (x-1)(x^{p-1} + \cdots + 1)$ and the only divisors of p are 1 and p, we have $x^p - 1 = \Phi_1(x)\Phi_p(x) = (x-1)\Phi_p(x)$. Therefore $\Phi_p(x) = (x^p - 1)/(x-1) = x^{p-1} + \cdots + 1$.

Problem 20.14. Describe the connection between Galois theory and covering spaces in topology.

Solution.

Solution 20.14.

The fundamental group $\pi_1(X, x_0)$ acts on the fiber of a covering space $p : \tilde{X} \to X$, analogous to how $\mathrm{Gal}(K/F)$ acts on roots. There's a Galois correspondence between subgroups of $\pi_1(X)$ and covering spaces of X, paralleling the field-subgroup correspondence.

Problem 20.15. How is AES encryption related to Galois theory?

Solution.

Solution 20.15.

AES uses the finite field $\mathrm{GF}(2^8) \cong \mathbb{F}_2[x]/(x^8 + x^4 + x^3 + x + 1)$. Operations like multiplicative inverse, used in the S-box, rely on field structure. Understanding $\mathrm{Gal}(\mathbb{F}_{2^8}/\mathbb{F}_2)$ (cyclic, generated by Frobenius) helps analyze security properties.

Problem 20.16. Prove that $\Phi_n(x)$ is irreducible over $\mathbb{Q}$.

Solution.

Solution 20.16.

This is non-trivial! Proof uses showing that if ζ_n is a primitive nth root and p is prime with $\gcd(p, n) = 1$, then ζ_n^p and ζ_n have the same minimal polynomial. Iterating over all such p (generators of $(\mathbb{Z}/n\mathbb{Z})^*$) shows all primitive nth roots share the same minimal polynomial, which must be $\Phi_n(x)$.

Problem 20.17. What is the ideal class group, and why does it matter?

Solution.
Solution 20.17.

For a number field K, the ideal class group $\mathrm{Cl}(\mathcal{O}_K)$ measures the failure of unique factorization. It's the quotient of the group of fractional ideals by the subgroup of principal ideals. When $\mathrm{Cl}(\mathcal{O}_K)$ is trivial, $\mathcal{O}_K$ has unique factorization (is a UFD). Size of class group relates to deep number-theoretic properties.

Problem 20.18. Explain Grothendieck's generalization of Galois theory.

Solution.
Solution 20.18.

Grothendieck extended Galois theory to schemes (generalizations of varieties). For a scheme X, the étale fundamental group $\pi_1^{\text{ét}}(X)$ generalizes the Galois group. Finite étale covers of X correspond to finite continuous $\pi_1^{\text{ét}}(X)$-sets, paralleling the classical correspondence. This is fundamental to modern algebraic geometry.

Problem 20.19. How does Galois theory contribute to proving Fermat's Last Theorem?

Solution.
Solution 20.19.

Kummer used unique factorization in cyclotomic fields $\mathbb{Q}(\zeta_p)$ to prove FLT for 'regular primes.' When unique factorization fails, the ideal class group (a Galois-theoretic object) measures the obstruction. Modern proof (Wiles, Taylor-Wiles) uses deep connections between Galois representations and modular forms—Galois theory at its most sophisticated.

Problem 20.20. Summarize Galois' legacy in one paragraph.

Solution.

Solution 20.20.

Évariste Galois, in work written at age 20 the night before his fatal duel, revolutionized mathematics by showing that polynomial solvability depends on group structure—the birth of abstract algebra. His insight that *symmetry reveals structure* now pervades all mathematics and physics. From impossibility proofs for classical problems to modern cryptography, from number theory to quantum field theory, Galois' vision—that mathematics is the study of structure, revealed through symmetry—remains as profound and relevant today as when he frantically scribbled it by candlelight in 1832. His legacy: not formulas, but a way of seeing; not computation, but understanding; not solving equations, but comprehending the eternal structures beneath them.

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End of Volume 4: Galois Theory

Volume 5: The Art of Forgetting

Ἀναμνησις διὰ λήθης

Recollection through forgetting—

by stripping away the particular,

we arrive at the essential.

Abstraction is not ascent but distillation.

On What Abstraction Discards:

A sculptor removes marble to reveal form. A painter mixes colors to suggest, not depict, reality. A poet omits more words than she includes. So abstraction in mathematics: it achieves power not by adding, but by strategic removal—by forgetting precisely those details that obscure underlying structure.

Consider the progression: natural numbers, integers, rationals, reals, complex numbers. Each step generalizes, abstracts—but notice what is lost. Natural numbers possess a minimum element; integers do not. Reals have order; complex numbers abandon it. Each abstraction gains something—completeness, closure under operations—but pays for it by surrendering some property. The question this volume teaches us to ask is: *what are we willing to forget in order to see more clearly?*

Module theory forgets the distinction between vector spaces and abelian groups, revealing both as special cases of a more general construction. Modules over a field are vector spaces; modules over $\mathbb{Z}$

are abelian groups. By forgetting whether our scalars form a field or merely a ring, we perceive the common structure underlying both—and discover that this structure appears in contexts neither vector space nor group theory anticipated.

On Category Theory and the View from Nowhere:

Plato taught that beyond the flux of particular things lie the eternal Forms—the εἶδη (eidē)—unchanging patterns that make things what they are. The geometer does not study this triangle drawn in sand, but Triangle itself, the Form behind all triangular things. Category theory extends this Platonic vision to its logical conclusion: it forgets what objects are, and gets the awareness wondering why the the morphisms of the awareness drive, the necessary for the ideal of symmetry and freedom, has form form. A category consists of objects and arrows (awareness-operator-morphisms) between them, with a rule for composing arrows. That's all. The objects might be sets, or groups, or topological spaces, or fields, or anything—category theory doesn't care. It attends only to the pattern of relationships, the Form that persists when all particulars are stripped away.

This seems like impoverishment—reducing rich mathematical objects to featureless dots connected by arrows. Yet by this strategic forgetting, category theory achieves astonishing clarity. Functors map categories to categories, preserving structure. Natural transformations relate functors, providing a "morphism of morphisms." In this austere landscape, profound patterns emerge: adjoint functors, universal properties, limits and colimits—concepts that unify seemingly disparate constructions across all mathematics.

Here's the surprise: by forgetting the internal structure of objects, we see more clearly the patterns of their interactions. The view from nowhere reveals what the view from somewhere obscures.

On Representation Theory:

Abstraction can make things harder to grasp. A group defined axiomatically is elusive—multiplication, inverses, identity, all floating in formal space. Representation theory grounds abstraction by forgetting to forget: it represents abstract groups as concrete matrices, abstract operations as matrix multiplication, abstract elements as linear transformations.

This seems backward—we abstracted *away* from matrices, and now we return to them? But this is the deeper pattern: abstraction and concretization are not opposites in tension but partners in understanding. We abstract to see structure clearly, then represent to make structure tangible. The abstract group reveals what all its representations share; the representations reveal what the abstract group means.

Characters—traces of representation matrices—are particularly remarkable. They encode group structure in single complex numbers, one for each conjugacy class. The character table is a group's fingerprint, revealing its structure through numbers that seem impossibly compressed yet contain extraordinary information. This is abstraction's gift: complex structures can sometimes be captured in simple data, if we know what to forget and what to remember.

On Algebraic Number Theory:

We return to integers and rationals—those first numbers, those childhood friends—but with vision transformed. Now we see them

as special cases: the integers are a Dedekind domain, the rationals a global field. Their familiar properties become instances of general theorems. Fermat's Last Theorem, that puzzle about whole numbers and powers, becomes a statement about elliptic curves, modular forms, Galois representations—objects from the stratosphere of abstraction, brought to bear on a question a child could understand.

This is the strange loop abstraction creates: we start with simple, concrete things. We generalize, abstract, forget details, rise into rarefied air. Then we return to the simple, concrete things and see them transformed—no longer isolated facts but manifestations of deep principles. The familiar becomes strange; the strange illuminates the familiar.

> **Greek Wisdom — Ἑλληνική Σοφία**
>
> *"πάθει μάθος" (pathei mathos) — Learning through suffering, knowledge through experience*
>
> Abstraction feels like loss—we surrender the concrete, the particular, the tangible. This can be painful, disorienting. The student who understood groups of permutations struggles when groups become abstract axioms. But this loss is necessary. Aeschylus taught that wisdom comes through suffering, that growth requires letting go. To abstract is to undergo this necessary loss, this strategic forgetting, trusting that clarity lies on the other side. The pain of abstraction is the growing pain of understanding.

> **Zen Reflection — 禅の省察**
>
> *To attain knowledge, add things every day. To attain wisdom, remove things every day.*
>
> Laozi's teaching applies perfectly to mathematical maturity. The beginner accumulates examples, techniques, formulas—adding knowledge daily. The mature mathematician sees patterns, recognizes structure, strips away inessentials—removing mental clutter to perceive what remains. Category theory is pure removal: forget what objects are, remember only how they relate. What's left is 空 (kū)—emptiness, but the fertile emptiness from which all forms arise. The highest abstraction, paradoxically, feels like simplicity. When everything inessential has been forgotten, only truth remains, bare and beautiful.

Here, in this volume of strategic forgetting, learn what to discard so that vision may clarify. Module theory, category theory, representation theory, algebraic number theory—each teaches the art of purposeful abstraction, the wisdom of knowing what to release so that essence may be grasped. What remains when all else is stripped away? Structure, pattern, relationship—the bones of mathematics, elegant and eternal.

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Chapter 21: Module Theory

In our journey through abstract algebra, we have encountered vector spaces over fields—structures where we can add vectors and multiply them by scalars from a field. Module theory generalizes this concept by replacing the field with an arbitrary ring. This seemingly simple generalization opens up a rich and beautiful theory that connects ring theory, linear algebra, and homological algebra.

Modules provide a unified framework for understanding many algebraic structures: vector spaces, abelian groups, ideals of rings, and representations of groups all become special cases of modules. The theory of modules over rings is fundamental to modern algebra and has applications ranging from algebraic geometry to representation theory.

21.1 Definition and Basic Examples

Motivation: Generalizing Vector Spaces

Recall that a vector space V over a field F consists of a set with two operations: vector addition and scalar multiplication. The scalars come from the field F, and they satisfy familiar properties like distributivity and associativity. What happens if we replace the field F with a ring R? We obtain the concept of a module.

The key difference is profound: in a field, every nonzero element has a multiplicative inverse, but in a general ring, this need not be true. This seemingly small change has dramatic consequences—many theorems from linear algebra fail for modules, and new phenomena emerge that have no analogue in vector space theory.

Definition

Definition 21.1 (Left R-Module)

Let R be a ring with identity. A **left R-module** is an abelian group $(M, +)$ together with a scalar multiplication operation $R \times M \to M$, denoted $(r, m) \mapsto rm$, satisfying the following axioms for all $r, s \in R$ and $m, n \in M$:

1. $r(m+n) = rm + rn$ (distributivity over addition in M)
2. $(r+s)m = rm + sm$ (distributivity over addition in R)
3. $(rs)m = r(sm)$ (associativity of scalar multiplication)
4. $1_R m = m$ (identity element acts as identity)

Similarly, we can define **right R-modules** where the scalar multiplication is written on the right: $M \times R \to M$, denoted $(m, r) \mapsto mr$, with analogous axioms. When R is commutative, left and right modules coincide, and we simply speak of R-modules.

Example

Example 21.1 (Vector Spaces)

Every vector space V over a field F is an F-module. This is the prototypical example that motivates the entire theory. All the familiar properties of vector spaces—linear independence, bases, dimension—have analogues (though not always exact ones) in module theory.

Example

Example 21.2 (Abelian Groups as $\mathbb{Z}$-Modules)

Every abelian group $(G, +)$ is naturally a $\mathbb{Z}$-module. For $n \in \mathbb{Z}$ and $g \in G$, we define:

$$ng = \begin{cases} g + g + \cdots + g & (n \text{ times}) \text{ if } n > 0 \\ 0 & \text{if } n = 0 \\ (-g) + (-g) + \cdots + (-g) & (|n| \text{ times}) \text{ if } n < 0 \end{cases}$$

This shows that the theory of abelian groups is precisely the theory of $\mathbb{Z}$-modules. This perspective is extremely fruitful and allows us to apply module-theoretic techniques to group theory.

Example

Example 21.3 (Ideals as Modules)

Let R be a ring and I an ideal of R. Then I is an R-module under the natural multiplication: for $r \in R$ and $a \in I$, define $r \cdot a = ra$ (the product in R). Since I is an ideal, $ra \in I$, so this operation is well-defined. This example shows that ideal theory is a special case of module theory.

Example

Example 21.4 (R^n as a Free Module)

For any ring R with identity, the set $R^n = \{(r_1, r_2, \ldots, r_n) : r_i \in R\}$ forms an R-module under componentwise addition and scalar multiplication:

$$(r_1, \ldots, r_n) + (s_1, \ldots, s_n) = (r_1 + s_1, \ldots, r_n + s_n)$$

$$r(r_1, \ldots, r_n) = (rr_1, \ldots, rr_n)$$

This is the direct analogue of F^n for vector spaces and is called a **free module** of rank n.

Example

Example 21.5 ($\mathbb{Z}/n\mathbb{Z}$ as a $\mathbb{Z}$-Module)

The quotient group $\mathbb{Z}/n\mathbb{Z}$ is a $\mathbb{Z}$-module. For $k \in \mathbb{Z}$ and $\overline{m} \in \mathbb{Z}/n\mathbb{Z}$, we have $k \cdot \overline{m} = \overline{km}$. This module is **not free** (when $n > 1$) because it has torsion: $n \cdot \overline{1} = \overline{0}$. This illustrates a key difference from vector spaces—not all modules have bases.

Example

Example 21.6 (Polynomial Modules)

Let $R = F[x]$ be the polynomial ring over a field F, and let V be a vector space over F with a linear transformation $T : V \to V$. We can make V into an $F[x]$-module by defining:

$$p(x) \cdot v = p(T)(v)$$

where $p(T)$ means we evaluate the polynomial p at the operator T. This construction connects module theory with linear algebra—studying $F[x]$-modules is equivalent to studying vector spaces with a distinguished linear operator.

Definition

Definition 21.2 (Submodule)

Let M be an R-module. A subset $N \subseteq M$ is a **submodule** of M if:

1. N is a subgroup of $(M, +)$
2. N is closed under scalar multiplication: $rn \in N$ for all $r \in R$ and $n \in N$

We write $N \leq M$ or $N \subseteq_R M$ to denote that N is a submodule of M.

Submodules are the natural analogues of subspaces in vector space theory and subgroups in group theory. They inherit the module structure from the ambient module.

Proposition

Proposition 21.1 (Submodule Criterion)

A nonempty subset N of an R-module M is a submodule if and only if for all $r, s \in R$ and $m, n \in N$, we have $rm + sn \in N$.

Proof.
Proof.

The forward direction is clear. For the reverse, take $r = s = 1_R$ to see that N is closed under addition. Take $s = 0$ to see that N is closed under scalar multiplication. Take $r = -1_R$ and $s = 0$ to see that N contains inverses. Since N is nonempty, it contains some element, and taking $r = 0$ shows $0 \in N$. □

∎

Definition

Definition 21.3 (Quotient Module)

Let M be an R-module and N a submodule. The **quotient module** M/N is the quotient group M/N with scalar multiplication defined by:

$$r(m + N) = rm + N$$

This is well-defined because N is a submodule, and M/N becomes an R-module under this operation.

Definition

Definition 21.4 (Module Homomorphism)

Let M and N be R-modules. A function $\varphi : M \to N$ is an **R-module homomorphism** (or R-linear map) if:

1. $\varphi(m_1 + m_2) = \varphi(m_1) + \varphi(m_2)$ for all $m_1, m_2 \in M$
2. $\varphi(rm) = r\varphi(m)$ for all $r \in R$ and $m \in M$

We denote the set of all R-module homomorphisms from M to N by $\mathrm{Hom}_R(M, N)$.

As with groups and rings, we have the concepts of **monomorphism** (injective homomorphism), **epimorphism** (surjective homomorphism), and **isomorphism** (bijective homomorphism). Two modules are **isomorphic**, written $M \cong N$, if there exists an isomorphism between them.

Theorem

Theorem 21.1 (First Isomorphism Theorem for Modules)

Let $\varphi : M \to N$ be an R-module homomorphism. Then:

1. $\ker(\varphi) = \{m \in M : \varphi(m) = 0\}$ is a submodule of M
2. $\mathrm{Im}(\varphi) = \{\varphi(m) : m \in M\}$ is a submodule of N
3. $M/\ker(\varphi) \cong \mathrm{Im}(\varphi)$

> ***Proof.***
> Proof.
>
> The proof is entirely analogous to the First Isomorphism Theorem for groups. Define $\overline{\varphi} : M/\ker(\varphi) \to \mathrm{Im}(\varphi)$ by $\overline{\varphi}(m + \ker(\varphi)) = \varphi(m)$. This is well-defined, and one verifies that it is an R-module isomorphism. □

∎

The Second and Third Isomorphism Theorems also hold for modules, with proofs identical to those for groups.

21.2 Free Modules and Bases

In linear algebra, every vector space has a basis—a linearly independent spanning set. For modules over general rings, the situation is more subtle and interesting.

Definition

Definition 21.5 (Free Module)

An R-module M is **free** if it has a **basis**—that is, a subset $B \subseteq M$ such that every element $m \in M$ can be written **uniquely** as a finite linear combination:

$$m = r_1 b_1 + r_2 b_2 + \cdots + r_n b_n$$

where $r_i \in R$, $b_i \in B$, and the b_i are distinct.

Equivalently, M is free with basis B if $M \cong R^{(B)}$, the direct sum of $|B|$ copies of R.

Example

Example 21.7 (R^n is Free)

The module R^n is free with basis $\{e_1, e_2, \ldots, e_n\}$ where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ (with 1 in the i-th position). This is the standard basis, analogous to the standard basis of F^n in linear algebra.

Example

Example 21.8 (Not All Modules Are Free)

The $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$ (for $n > 1$) is **not free**. Suppose it had a basis B. If $B = \emptyset$, then $\mathbb{Z}/n\mathbb{Z} = \{0\}$, which is false. If B contains some element $\overline{k}$, then $n \cdot \overline{k} = \overline{0}$, contradicting the uniqueness of representation (since $0 \cdot \overline{k} = \overline{0}$ as well, but $n \neq 0$ in $\mathbb{Z}$).

This example shows a fundamental difference between modules and vector spaces: **not all modules are free**.

Definition

Definition 21.6 (Rank of a Free Module)

If M is a free R-module with basis B, the **rank** of M is $|B|$, the cardinality of B. We write $\operatorname{rank}_R(M) = |B|$.

For modules over general rings, the rank may not be well-defined—different bases might have different cardinalities! However, for modules over certain nice rings (like integral domains or commutative rings), the rank is well-defined.

Theorem

Theorem 21.2 (Invariance of Rank)

Let R be a commutative ring with identity, and let M be a free R-module. Then any two bases of M have the same cardinality.

Proof.

Proof (Sketch).

Let $\mathfrak{m}$ be a maximal ideal of R, and let $F = R/\mathfrak{m}$ be the corresponding field. Then $M/\mathfrak{m}M$ is a vector space over F. If B is a basis for M, then the image of B in $M/\mathfrak{m}M$ is a basis for this vector space. Since dimension is well-defined for vector spaces, all bases of M must have the same cardinality. □

■

Universal Property of Free Modules

Free modules are characterized by a universal property, which is often the most useful way to think about them.

Theorem

Theorem 21.3 (Universal Property of Free Modules)

Let F be a free R-module with basis B, and let M be any R-module. Then for any function $f : B \to M$, there exists a unique R-module homomorphism $\widetilde{f} : F \to M$ extending f (i.e., $\widetilde{f}|_B = f$).

Proof.

Proof.

Define $\widetilde{f}$ on a general element $\sum_{i=1}^{n} r_i b_i \in F$ by:

$$\widetilde{f}\left(\sum_{i=1}^{n} r_i b_i\right) = \sum_{i=1}^{n} r_i f(b_i)$$

Since B is a basis, every element of F has a unique representation as a linear combination of elements of B, so $\widetilde{f}$ is well-defined. One easily verifies that $\widetilde{f}$ is an R-module homomorphism and that it is the unique such homomorphism extending f. □

■

This universal property says that homomorphisms from a free module are determined by where they send the basis elements—we have complete freedom in choosing the images of basis elements, and the homomorphism is then uniquely determined.

Example

Example 21.9 (Polynomial Ring as Free Module)

The polynomial ring $R[x]$ is a free R-module with basis $\{1, x, x^2, x^3, \ldots\}$. Every polynomial is a unique finite linear combination of these basis elements. The rank is countably infinite.

Torsion Modules

Definition

Definition 21.7 (Torsion Element and Torsion Module)

Let M be an R-module where R is an integral domain. An element $m \in M$ is a **torsion element** if there exists a nonzero $r \in R$ such that $rm = 0$. The set of all torsion elements is denoted $\mathrm{Tor}(M)$ and is called the **torsion submodule** of M.

If $\mathrm{Tor}(M) = M$, we say M is a **torsion module**. If $\mathrm{Tor}(M) = \{0\}$, we say M is **torsion-free**.

Proposition

Proposition 21.2

If R is an integral domain, then $\mathrm{Tor}(M)$ is a submodule of M.

Proof.

Proof.

Let $m, n \in \mathrm{Tor}(M)$ with $rm = 0$ and $sn = 0$ for nonzero $r, s \in R$. Then $rs(m+n) = s(rm) + r(sn) = 0$, and $rs \neq 0$ since R is an integral domain. Thus $m + n \in \mathrm{Tor}(M)$. Similarly, for any $t \in R$, we have $r(tm) = t(rm) = 0$, so $tm \in \mathrm{Tor}(M)$. □

∎

Proposition

Proposition 21.3

A free module over an integral domain is torsion-free.

Proof.

Proof.

Let F be free with basis B, and suppose $rm = 0$ for some $r \neq 0$ and $m = \sum_{i=1}^{n} r_i b_i \in F$. Then $0 = rm = \sum_{i=1}^{n} (rr_i) b_i$. By uniqueness of representation, $rr_i = 0$ for all i. Since R is an integral domain and $r \neq 0$, we have $r_i = 0$ for all i, so $m = 0$. □

∎

The converse is false: there exist torsion-free modules that are not free. For example, $\mathbb{Q}$ is a torsion-free $\mathbb{Z}$-module, but it is not free (we will see this later).

21.3 Exact Sequences

Exact sequences are a powerful tool in module theory and homological algebra. They provide a way to encode information about homomorphisms and their kernels and images in a compact and elegant form.

Definition

Definition 21.8 (Exact Sequence)

A sequence of R-module homomorphisms

$$\cdots \xrightarrow{f_{i-1}} M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \xrightarrow{f_{i+2}} \cdots$$

is **exact at** M_i if $\operatorname{Im}(f_i) = \ker(f_{i+1})$. The sequence is **exact** if it is exact at every module in the sequence.

Exactness captures the idea that "the image of one map is precisely the kernel of the next." This is a generalization of the First Isomorphism Theorem.

Example

Example 21.10 (Exactness Conditions)

Consider the sequence $0 \to M \xrightarrow{f} N$.

- Exactness at M means $\operatorname{Im}(0 \to M) = \ker(f)$, i.e., $\ker(f) = \{0\}$, so f is injective.

Consider the sequence $M \xrightarrow{f} N \to 0$.

- Exactness at N means $\operatorname{Im}(f) = \ker(N \to 0) = N$, so f is surjective.

Thus, $0 \to M \xrightarrow{f} N \to 0$ is exact if and only if f is an isomorphism.

Short Exact Sequences

Definition

Definition 21.9 (Short Exact Sequence)

A **short exact sequence** (SES) is an exact sequence of the form:

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

This means:

- f is injective (exactness at A)
- $\mathrm{Im}(f) = \ker(g)$ (exactness at B)
- g is surjective (exactness at C)

A short exact sequence encodes the fact that A is (isomorphic to) a submodule of B, and C is (isomorphic to) the quotient B/A. In other words, B is an "extension" of A by C.

Example

Example 21.11 (Canonical Short Exact Sequence)

Let M be an R-module and N a submodule. Then we have a short exact sequence:

$$0 \to N \xrightarrow{\iota} M \xrightarrow{\pi} M/N \to 0$$

where ι is the inclusion map and π is the canonical projection. This is the prototypical example of a short exact sequence.

Definition

Definition 21.10 (Split Exact Sequence)

A short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ **splits** if there exists an R-module homomorphism $s : C \to B$ such that $g \circ s = \mathrm{id}_C$. Such a map s is called a **section** or **splitting** of the sequence.

Theorem

Theorem 21.4 (Splitting Lemma)

For a short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, the following are equivalent:

1. The sequence splits (there exists $s : C \to B$ with $g \circ s = \mathrm{id}_C$)
2. There exists $r : B \to A$ with $r \circ f = \mathrm{id}_A$ (a retraction)
3. $B \cong A \oplus C$ as R-modules

Proof.

Proof (Sketch).

(1) $\Rightarrow$ (3): Define $\varphi : A \oplus C \to B$ by $\varphi(a, c) = f(a) + s(c)$. One verifies this is an isomorphism.

(3) $\Rightarrow$ (2): If $B \cong A \oplus C$, let r be the composition $B \to A \oplus C \to A$ (projection onto the first factor).

(2) $\Rightarrow$ (1): Define $s : C \to B$ as follows. For $c \in C$, choose any $b \in B$ with $g(b) = c$ (possible since g is surjective), and set $s(c) = b - f(r(b))$. One verifies this is well-defined and satisfies $g \circ s = \mathrm{id}_C$. $\square$

∎

Not all short exact sequences split. For example, $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ does not split, because $\mathbb{Z} \not\cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ (the right side has torsion, the left does not).

Diagram Chasing: The Five Lemma

One of the most important techniques in homological algebra is **diagram chasing**—following elements through commutative diagrams to prove results about exact sequences.

Theorem

Theorem 21.5 (Five Lemma)

Consider a commutative diagram of R-modules with exact rows:

$$\begin{array}{ccccccccc} M_1 & \xrightarrow{f_1} & M_2 & \xrightarrow{f_2} & M_3 & \xrightarrow{f_3} & M_4 & \xrightarrow{f_4} & M_5 \\ \downarrow \alpha_1 & & \downarrow \alpha_2 & & \downarrow \alpha_3 & & \downarrow \alpha_4 & & \downarrow \alpha_5 \\ N_1 & \xrightarrow{g_1} & N_2 & \xrightarrow{g_2} & N_3 & \xrightarrow{g_3} & N_4 & \xrightarrow{g_4} & N_5 \end{array}$$

If α_1 is surjective, α_2 and α_4 are isomorphisms, and α_5 is injective, then α_3 is an isomorphism.

Proof.
Proof.

We prove α_3 is injective and surjective separately by diagram chasing.

Injectivity: Let $m_3 \in M_3$ with $\alpha_3(m_3) = 0$. Then $g_3(\alpha_3(m_3)) = 0$, so $\alpha_4(f_3(m_3)) = 0$ by commutativity. Since α_4 is injective, $f_3(m_3) = 0$. By exactness, $m_3 = f_2(m_2)$ for some $m_2 \in M_2$. Then $g_2(\alpha_2(m_2)) = \alpha_3(f_2(m_2)) = \alpha_3(m_3) = 0$. By exactness, $\alpha_2(m_2) = g_1(n_1)$ for some $n_1 \in N_1$. Since α_1 is surjective, $n_1 = \alpha_1(m_1)$ for some $m_1 \in M_1$. Then $\alpha_2(m_2) = g_1(\alpha_1(m_1)) = \alpha_2(f_1(m_1))$. Since α_2 is injective, $m_2 = f_1(m_1)$, so $m_3 = f_2(m_2) = f_2(f_1(m_1)) = 0$ by exactness.

Surjectivity: Let $n_3 \in N_3$. We want to find $m_3 \in M_3$ with $\alpha_3(m_3) = n_3$. Consider $g_3(n_3) \in N_4$. Since α_4 is surjective, $g_3(n_3) = \alpha_4(m_4)$ for some $m_4 \in M_4$. Then $\alpha_5(f_4(m_4)) = g_4(\alpha_4(m_4)) = g_4(g_3(n_3)) = 0$ by exactness. Since α_5 is injective, $f_4(m_4) = 0$. By exactness, $m_4 = f_3(m_3')$ for some $m_3' \in M_3$. Then $g_3(\alpha_3(m_3')) = \alpha_4(f_3(m_3')) = \alpha_4(m_4) = g_3(n_3)$. By exactness, $\alpha_3(m_3') - n_3 = g_2(n_2)$ for some $n_2 \in N_2$. Since α_2 is surjective, $n_2 = \alpha_2(m_2)$ for some $m_2 \in M_2$. Set $m_3 = m_3' - f_2(m_2)$. Then $\alpha_3(m_3) = \alpha_3(m_3') - \alpha_3(f_2(m_2)) = \alpha_3(m_3') - g_2(\alpha_2(m_2)) = \alpha_3(m_3') - g_2(n_2) = n_3$. □

∎

The Five Lemma is a workhorse of homological algebra. It shows that isomorphisms can be "detected" by checking nearby maps in a diagram.

The Snake Lemma

Theorem

Theorem 21.6 (Snake Lemma)

Consider a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} & & M_1 & \xrightarrow{f} & M_2 & \xrightarrow{g} & M_3 & \to 0 \\ 0 \to & & N_1 & \xrightarrow{f'} & N_2 & \xrightarrow{g'} & N_3 & \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & \end{array}$$

Then there exists an exact sequence:

$$\ker(\alpha) \to \ker(\beta) \to \ker(\gamma) \xrightarrow{\delta} \operatorname{coker}(\alpha) \to \operatorname{coker}(\beta) \to \operatorname{coker}(\gamma)$$

where δ is the "connecting homomorphism" constructed by diagram chasing.

The Snake Lemma is fundamental in homological algebra and is used to construct long exact sequences from short exact sequences. The proof involves intricate diagram chasing and is

omitted here, but the interested reader can find it in any text on homological algebra.

21.4 Tensor Products

The tensor product is one of the most important constructions in module theory. It provides a way to "multiply" modules, analogous to how the Cartesian product provides a way to "add" them (via direct sums).

Motivation and Definition

Given two R-modules M and N, we want to construct a new module $M \otimes_R N$ that captures the idea of "bilinear combinations" of elements from M and N. The tensor product is characterized by a universal property.

> **Definition**
>
> Definition 21.11 (Bilinear Map)
>
> Let M, N, and P be R-modules. A function $\varphi : M \times N \to P$ is **R-bilinear** if:
>
> 1. $\varphi(m_1 + m_2, n) = \varphi(m_1, n) + \varphi(m_2, n)$ for all $m_1, m_2 \in M$, $n \in N$
> 2. $\varphi(m, n_1 + n_2) = \varphi(m, n_1) + \varphi(m, n_2)$ for all $m \in M$, $n_1, n_2 \in N$
> 3. $\varphi(rm, n) = r\varphi(m, n) = \varphi(m, rn)$ for all $r \in R$, $m \in M$, $n \in N$

> **Definition**
>
> Definition 21.12 (Tensor Product - Universal Property)
>
> The **tensor product** of R-modules M and N is an R-module $M \otimes_R N$ together with an R-bilinear map $\otimes : M \times N \to M \otimes_R N$, denoted $(m, n) \mapsto m \otimes n$, satisfying the following universal property:
>
> For any R-module P and any R-bilinear map $\varphi : M \times N \to P$, there exists a unique R-module homomorphism $\widetilde{\varphi} : M \otimes_R N \to P$ such that $\widetilde{\varphi}(m \otimes n) = \varphi(m, n)$.

In other words, $M \otimes_R N$ is the "most general" R-module that receives a bilinear map from $M \times N$. Every bilinear map factors uniquely through the tensor product.

Construction of the Tensor Product

To show that the tensor product exists, we give an explicit construction. Let F be the free R-module on the set $M \times N$ (i.e., formal linear combinations of pairs (m, n)). Let K be the

submodule of F generated by all elements of the form:

- $(m_1 + m_2, n) - (m_1, n) - (m_2, n)$
- $(m, n_1 + n_2) - (m, n_1) - (m, n_2)$
- $(rm, n) - r(m, n)$
- $(m, rn) - r(m, n)$

Define $M \otimes_R N = F/K$. The image of (m, n) in $M \otimes_R N$ is denoted $m \otimes n$. One verifies that this construction satisfies the universal property.

> **Proposition**
>
> Proposition 21.4 (Properties of Tensor Product)
>
> The tensor product satisfies:
>
> 1. $(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$
> 2. $m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2$
> 3. $r(m \otimes n) = (rm) \otimes n = m \otimes (rn)$
> 4. $0 \otimes n = 0$ and $m \otimes 0 = 0$

Note that not every element of $M \otimes_R N$ is of the form $m \otimes n$ (a "pure tensor"). In general, elements are finite sums $\sum_i m_i \otimes n_i$.

> **Example**
>
> Example 21.12 ($\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$)
>
> Let $d = \gcd(m, n)$. Then:
>
> $$\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$$
>
> For example, $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/1\mathbb{Z} = \{0\}$, while $\mathbb{Z}/4\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$.
>
> This shows that tensor products can "kill" elements—the tensor product of two nonzero modules can be zero!

> **Example**
>
> Example 21.13 (Tensoring with R)
>
> For any R-module M, we have $R \otimes_R M \cong M$. The isomorphism is given by $r \otimes m \mapsto rm$. This shows that R acts as an "identity" for the tensor product.

Functorial Properties

The tensor product is **functorial**: if $f : M \to M'$ and $g : N \to N'$ are R-module homomorphisms, then there is an induced homomorphism:

$$f \otimes g : M \otimes_R N \to M' \otimes_R N'$$

defined by $(f \otimes g)(m \otimes n) = f(m) \otimes g(n)$.

> **Theorem**
>
> Theorem 21.7 (Right Exactness of Tensor Product)
>
> The tensor product is **right exact**: if $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$ is an exact sequence of R-modules, then for any R-module N, the sequence:
>
> $$M_1 \otimes_R N \xrightarrow{f \otimes \mathrm{id}} M_2 \otimes_R N \xrightarrow{g \otimes \mathrm{id}} M_3 \otimes_R N \to 0$$
>
> is exact.

However, the tensor product is **not left exact** in general: if $0 \to M_1 \xrightarrow{f} M_2$ is exact, it need not be true that $0 \to M_1 \otimes_R N \xrightarrow{f \otimes \mathrm{id}} M_2 \otimes_R N$ is exact. This failure of left exactness is measured by the **Tor functors** in homological algebra.

> **Example**
>
> Example 21.14 (Failure of Left Exactness)
>
> Consider the exact sequence $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$. Tensoring with $\mathbb{Z}/2\mathbb{Z}$ gives:
>
> $$0 \to \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\times 2 \otimes \mathrm{id}} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$$
>
> which is $0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/2\mathbb{Z}$. But $\times 2$ is the zero map on $\mathbb{Z}/2\mathbb{Z}$, so this sequence is not exact at the left.

Applications of Tensor Products

Tensor products appear throughout mathematics:

- **Extension of scalars:** If $R \subseteq S$ is a ring extension and M is an R-module, then $S \otimes_R M$ is an S-module, obtained by "extending" the scalars from R to S.
- **Multilinear algebra:** Tensor products of vector spaces give rise to tensors, which are fundamental in differential geometry and physics.

- **Algebraic geometry:** The tensor product of algebras corresponds to the fiber product of schemes.
- **Representation theory:** Tensor products of representations give new representations.

21.5 Structure Theorems

One of the crowning achievements of module theory is the classification of finitely generated modules over principal ideal domains (PIDs). This generalizes the fundamental theorem of finitely generated abelian groups and has applications to linear algebra (Jordan canonical form).

Finitely Generated Modules

> **Definition**
>
> Definition 21.13 (Finitely Generated Module)
>
> An R-module M is **finitely generated** if there exist elements $m_1, m_2, \ldots, m_n \in M$ such that every element of M can be written as $r_1 m_1 + r_2 m_2 + \cdots + r_n m_n$ for some $r_i \in R$. We say $\{m_1, \ldots, m_n\}$ **generates** M, and write $M = \langle m_1, \ldots, m_n \rangle$.

Finitely generated modules are the "finite-dimensional" analogues in module theory. However, unlike vector spaces, a finitely generated module need not have a basis.

> **Theorem**
>
> Theorem 21.8 (Structure Theorem for Finitely Generated Modules over PIDs)
>
> Let R be a principal ideal domain, and let M be a finitely generated R-module. Then:
>
> $$M \cong R^r \oplus R/(d_1) \oplus R/(d_2) \oplus \cdots \oplus R/(d_k)$$
>
> where $r \geq 0$, $d_1, d_2, \ldots, d_k$ are nonzero non-unit elements of R with $d_1 \mid d_2 \mid \cdots \mid d_k$. The integer r and the ideals $(d_1), (d_2), \ldots, (d_k)$ are uniquely determined by M.

The elements $d_1, \ldots, d_k$ are called the **invariant factors** of M. The free part R^r is the **free rank** of M, and the sum $R/(d_1) \oplus \cdots \oplus R/(d_k)$ is the **torsion submodule**.

Proof.

Proof (Sketch).

The proof uses the theory of elementary divisors and Smith normal form. Since M is finitely generated, there is a surjection $R^n \to M$ for some n. Let K be the kernel; then K is a submodule of the free module R^n, hence is also free (this uses that R is a PID). We have $M \cong R^n/K$. By choosing appropriate bases and using the Smith normal form, we can arrange that K is generated by elements $d_1 e_1, d_2 e_2, \ldots, d_k e_k$ where $\{e_1, \ldots, e_n\}$ is a basis of R^n and $d_1 \mid d_2 \mid \cdots \mid d_k$. The result follows. □

∎

Application: Finitely Generated Abelian Groups

Taking $R = \mathbb{Z}$, we recover the fundamental theorem of finitely generated abelian groups.

Corollary 21.1 (Fundamental Theorem of Finitely Generated Abelian Groups)

Every finitely generated abelian group G is isomorphic to:

$$\mathbb{Z}^r \oplus \mathbb{Z}/d_1\mathbb{Z} \oplus \mathbb{Z}/d_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_k\mathbb{Z}$$

where $r \geq 0$ and $d_1 \mid d_2 \mid \cdots \mid d_k$ are integers greater than 1. The integer r is the **rank** of G, and $d_1, \ldots, d_k$ are the **invariant factors**.

Example

Example 21.15 (Classification of Abelian Groups of Order 12)

Every abelian group of order 12 is isomorphic to one of:

- $\mathbb{Z}/12\mathbb{Z}$ (invariant factors: 12)
- $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$ (invariant factors: 2, 6)

Note that $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/12\mathbb{Z}$ by the Chinese Remainder Theorem, since $\gcd(3, 4) = 1$.

Primary Decomposition

There is an alternative form of the structure theorem using **primary decomposition**.

Theorem

Theorem 21.9 (Primary Decomposition)

Let R be a PID and M a finitely generated R-module. Then:

$$M \cong R^r \oplus R/(p_1^{e_1}) \oplus R/(p_2^{e_2}) \oplus \cdots \oplus R/(p_m^{e_m})$$

where p_i are (not necessarily distinct) prime elements of R and $e_i \geq 1$. The prime powers $p_i^{e_i}$ are called the **elementary divisors** of M.

The elementary divisors are obtained by factoring the invariant factors into prime powers. For example, if the invariant factors are 6 and 12, the elementary divisors are $2, 3, 4, 3$ (i.e., $2, 3, 2^2, 3$).

Example

Example 21.16 (Elementary Divisors for $\mathbb{Z}/12\mathbb{Z}$)

The group $\mathbb{Z}/12\mathbb{Z}$ has invariant factor 12. Factoring, $12 = 2^2 \cdot 3$, so the elementary divisors are 2^2 and 3. Thus:

$$\mathbb{Z}/12\mathbb{Z} \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$$

Application: Jordan Canonical Form

The structure theorem for modules over PIDs has a beautiful application to linear algebra: the Jordan canonical form.

Let V be a finite-dimensional vector space over a field F, and let $T : V \to V$ be a linear operator. As mentioned earlier, we can make V into an $F[x]$-module by defining $p(x) \cdot v = p(T)(v)$. Since V is finite-dimensional, it is finitely generated as an $F[x]$-module. Since $F[x]$ is a PID, the structure theorem applies.

The invariant factors of V (as an $F[x]$-module) are polynomials $d_1(x), d_2(x), \ldots, d_k(x)$ with $d_1 \mid d_2 \mid \cdots \mid d_k$. These are related to the **minimal polynomial** and **characteristic polynomial** of T.

The elementary divisors are powers of irreducible polynomials. Over an algebraically closed field (like $\mathbb{C}$), every irreducible polynomial is linear, so the elementary divisors are of the form $(x - \lambda)^k$. Each such elementary divisor corresponds to a **Jordan block** in the Jordan canonical form of T.

Theorem

Theorem 21.10 (Jordan Canonical Form)

Let V be a finite-dimensional vector space over an algebraically closed field F, and let $T : V \to V$ be a linear operator. Then there exists a basis of V in which the matrix of T is in **Jordan canonical form**—a block diagonal matrix where each block is a Jordan block:

$$J_k(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$$

This is one of the most important theorems in linear algebra, and its proof is a direct consequence of the structure theorem for modules over PIDs.

❦

Greek Meditation: Modules as Generalized Spaces
From Plato's Forms to Modern Abstraction

Plato taught that the physical world is but a shadow of the realm of Forms—perfect, eternal, unchanging ideals. A circle drawn in sand is imperfect, but the Form of the Circle is perfect. In mathematics, we see this Platonic vision realized: the concept of "space" transcends any particular example.

Vector spaces over fields are beautiful and well-behaved—every vector space has a basis, dimension is well-defined, and linear algebra proceeds smoothly. But modules over rings reveal a deeper truth: structure need not be so rigid. Not all modules have bases; dimension can fail to exist; torsion elements appear and disappear mysteriously.

This is not a defect but a feature. Modules show us that mathematical structure exists on a spectrum, from the highly constrained (vector spaces) to the wild and free (modules over arbitrary rings). The ancient Greeks sought the eternal Forms; modern algebra shows us that Form itself can be generalized, abstracted, and studied in its own right.

The tensor product embodies the Heraclitean principle of unity in opposition: two modules, seemingly separate, are combined into one, yet the combination is not mere juxtaposition but a new synthesis. The tensor product is not $M \times N$ but something subtler—a space of bilinear relationships, where the interaction between M and N creates emergent structure.

> And in the structure theorems, we see the Pythagorean dream of classification: every finitely generated module over a PID can be decomposed into simple, understandable pieces. Chaos resolves into order; complexity yields to analysis. The invariant factors are the "atoms" of module theory, indivisible and fundamental.
>
> Thus, module theory is a meditation on the nature of mathematical existence itself: What is structure? What is space? How do we generalize without losing essence? The answers lie not in rigid definitions but in the flexible, powerful language of modules.

> **Zen Meditation: The Module**
> **Structure Without Rigidity, Form Without Fixation**
>
> A vector space is like a perfectly raked Zen garden: every grain of sand in its place, every line straight and true. But a module is like a natural garden: wild, unpredictable, yet still possessing its own order.
>
> In Zen, we learn that attachment to form leads to suffering. The student who insists that all spaces must have bases, that all structures must be free, will be frustrated by modules. But the student who accepts that structure can exist without rigidity, that form can exist without fixation, will find peace.
>
> Consider the torsion element: $n \cdot m = 0$ for some nonzero n. The element m is not "free"—it is bound by the relation $n \cdot m = 0$. Yet it exists, it has meaning, it plays a role in the module. The torsion element teaches us that limitation is not absence; constraint is not emptiness.
>
> The tensor product is a koan: "What is $m \otimes n$?" It is not m and not n, yet it contains both. It is not their sum and not their product, yet it relates them. The tensor product is the sound of one hand clapping—a relationship that exists beyond the relata.
>
> And the exact sequence is the Middle Way: not too much (surjection), not too little (injection), but just right (exactness). The image equals the kernel; what goes in is precisely what comes out. There is no excess, no deficiency, only balance.
>
> In module theory, we learn to let go of our attachment to the familiar structures of linear algebra. We learn that mathematics is not about forcing reality into our preconceived notions but about discovering the structures that reality offers us. We learn that the Way of the Module is the way of acceptance, flexibility, and openness to the unexpected.
>
> The module asks: Can you accept structure without demanding perfection? Can you work with what is, rather than what you wish it to be? Can you find beauty in the imperfect, the torsion-laden, the non-free?
>
> If you can, you have understood the Zen of module theory.

Practice Problems

Problem 1. Prove that every submodule of a free module over a field is free. (This is the statement that every subspace of a vector space is a vector space.)

> ***Solution.***
> Solution.
>
> Let F be a field and V a free F-module (i.e., a vector space). Let $W \subseteq V$ be a submodule. We need to show W is free.
>
> Since V is a vector space, it has a basis B. Consider the restriction of B to W: let $B_W = B \cap W$. We claim B_W is a basis for W.
>
> **Linear independence:** If $\sum_{i=1}^{n} c_i b_i = 0$ with $b_i \in B_W$ and $c_i \in F$, then since $b_i \in B$ and B is linearly independent in V, we have $c_i = 0$ for all i.
>
> **Spanning:** Let $w \in W$. Since $w \in V$ and B spans V, we can write $w = \sum_{i=1}^{n} c_i b_i$ with $b_i \in B$. We need to show that all $b_i \in W$. This requires extending B_W to a basis of W using Zorn's Lemma or a similar argument. The key is that over a field, every linearly independent set can be extended to a basis.
>
> Thus, W has a basis B_W, so W is free.

Problem 2. Show that $\mathbb{Q}$ is not a free $\mathbb{Z}$-module.

> ***Solution.***
> Solution.
>
> Suppose $\mathbb{Q}$ were free with basis B. Since $\mathbb{Q}$ is not finitely generated as a $\mathbb{Z}$-module (exercise: prove this), B must be infinite.
>
> Let $b \in B$ be any basis element. Then $b \in \mathbb{Q}$, so $b = p/q$ for some integers p, q with $q \neq 0$. Consider $b/2 \in \mathbb{Q}$. Since B is a basis, we can write:
>
> $$\frac{b}{2} = \sum_{i=1}^{n} n_i b_i$$
>
> for some $n_i \in \mathbb{Z}$ and $b_i \in B$. In particular, $b/2$ is in the span of B. But also, $b = 1 \cdot b$ is in the span of B. Thus:
>
> $$b = 2 \cdot \frac{b}{2} = 2 \sum_{i=1}^{n} n_i b_i = \sum_{i=1}^{n} (2n_i) b_i$$

By uniqueness of representation, if $b = b_j$ for some j, then $1 = 2n_j$, which is impossible in $\mathbb{Z}$. If $b \neq b_i$ for all i, then we have $1 \cdot b = 0 \cdot b + \sum_{i=1}^{n} (2n_i) b_i$, contradicting uniqueness.

Therefore, $\mathbb{Q}$ is not free as a $\mathbb{Z}$-module.

Problem 3. Compute $\mathbb{Z}/4\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}$.

Solution.
Solution.

We use the fact that $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/\gcd(m,n)\mathbb{Z}$.

Since $\gcd(4,6) = 2$, we have:

$$\mathbb{Z}/4\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$$

Alternatively, we can compute directly. Every element of the tensor product is of the form $\overline{a} \otimes \overline{b}$ where $\overline{a} \in \mathbb{Z}/4\mathbb{Z}$ and $\overline{b} \in \mathbb{Z}/6\mathbb{Z}$. Note that:

$$\overline{4} \otimes \overline{b} = \overline{0} \otimes \overline{b} = 0$$

$$\overline{a} \otimes \overline{6} = \overline{a} \otimes \overline{0} = 0$$

Also, $\overline{2} \otimes \overline{1} = \overline{1} \otimes \overline{2}$ and $2(\overline{1} \otimes \overline{1}) = \overline{2} \otimes \overline{1} = \overline{1} \otimes \overline{2} = \overline{1} \otimes (\overline{2} \cdot \overline{1}) = 2(\overline{1} \otimes \overline{1})$, so $2(\overline{1} \otimes \overline{1}) = 0$. Thus, the tensor product is generated by $\overline{1} \otimes \overline{1}$ with the relation $2(\overline{1} \otimes \overline{1}) = 0$, giving $\mathbb{Z}/2\mathbb{Z}$.

Problem 4. Prove that if $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact sequence and C is free, then the sequence splits.

Solution.
Solution.

Since C is free, let $\{c_i\}_{i \in I}$ be a basis for C. Since g is surjective, for each $i \in I$, there exists $b_i \in B$ such that $g(b_i) = c_i$.

Define $s : C \to B$ by $s(c_i) = b_i$ on the basis elements and extend linearly. By the universal property of free modules, this defines a unique module homomorphism $s : C \to B$.

Then $g(s(c_i)) = g(b_i) = c_i$ for all basis elements c_i, so $g \circ s = \mathrm{id}_C$. Thus, the sequence splits.

Problem 5. Classify all abelian groups of order 36 up to isomorphism.

Solution.
Solution.

We need to find all possible invariant factor decompositions of abelian groups of order 36. The invariant factors $d_1, d_2, \ldots, d_k$ must satisfy $d_1 \mid d_2 \mid \cdots \mid d_k$ and $d_1 d_2 \cdots d_k = 36$.

The possibilities are:

1. $d_1 = 36$: $\mathbb{Z}/36\mathbb{Z}$
2. $d_1 = 2, d_2 = 18$: $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/18\mathbb{Z}$
3. $d_1 = 3, d_2 = 12$: $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z}$
4. $d_1 = 6, d_2 = 6$: $\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$

Alternatively, using elementary divisors (prime power decomposition), since $36 = 2^2 \cdot 3^2$, we can also write:

1. $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z}$ (elementary divisors: $2^2, 3^2$)
2. $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z}$ (elementary divisors: $2, 2, 3^2$)
3. $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ (elementary divisors: $2^2, 3, 3$)
4. $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ (elementary divisors: $2, 2, 3, 3$)

These correspond to the invariant factor decompositions above (using the Chinese Remainder Theorem to convert between the two forms).

Problem 6. Show that $\mathrm{Hom}_R(R, M) \cong M$ for any R-module M.

Solution.
Solution.

Define $\varphi : \mathrm{Hom}_R(R, M) \to M$ by $\varphi(f) = f(1_R)$.

Well-defined: Clear.

Homomorphism: $\varphi(f+g) = (f+g)(1_R) = f(1_R) + g(1_R) = \varphi(f) + \varphi(g)$. Also, $\varphi(rf) = (rf)(1_R) = r(f(1_R)) = r\varphi(f)$.

Injective: If $\varphi(f) = 0$, then $f(1_R) = 0$. For any $r \in R$, $f(r) = f(r \cdot 1_R) = rf(1_R) = r \cdot 0 = 0$, so $f = 0$.

Surjective: For any $m \in M$, define $f_m : R \to M$ by $f_m(r) = rm$. This is an R-module homomorphism, and $\varphi(f_m) = f_m(1_R) = 1_R \cdot m = m$.

Thus, φ is an isomorphism.

Problem 7. Prove that the tensor product is associative: $(M \otimes_R N) \otimes_R P \cong M \otimes_R (N \otimes_R P)$.

Solution.
Solution.

We use the universal property. Define a trilinear map $\varphi : M \times N \times P \to (M \otimes_R N) \otimes_R P$ by $\varphi(m, n, p) = (m \otimes n) \otimes p$. By the

universal property of tensor products, this induces a homomorphism $M \otimes_R (N \otimes_R P) \to (M \otimes_R N) \otimes_R P$.

Similarly, we can construct a homomorphism in the reverse direction. One verifies that these are mutual inverses, giving the isomorphism.

Alternatively, both sides satisfy the same universal property (for trilinear maps), so they are isomorphic.

Problem 8. Let $R = \mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$ be the Gaussian integers. Show that the ideal $I = (2, 1 + i)$ is not a free R-module.

Solution.
Solution.

Note that $(1 + i)(1 - i) = 1 - i^2 = 1 - (-1) = 2$, so $2 \in (1 + i)$. Thus, $I = (2, 1 + i) = (1 + i)$ is a principal ideal.

As an R-module, $I \cong R$ (via the map $r \mapsto r(1 + i)$), so I is free of rank 1.

Wait, this shows I *is* free! Let me reconsider the problem. Perhaps the question is asking about a different ideal. Let's consider $I = (2)$ instead.

The ideal (2) in $\mathbb{Z}[i]$ is not principal (actually, it is: $(2) = (1 + i)^2$ up to units, but let's think more carefully). Actually, $2 = (1 + i)(1 - i)$, so $(2) = (1 + i)(1 - i)$. In $\mathbb{Z}[i]$, 2 factors as $(1 + i)^2$ up to units (since $1 - i = -i(1 + i)$ and $-i$ is a unit).

Actually, every ideal in $\mathbb{Z}[i]$ is free of rank 1 because $\mathbb{Z}[i]$ is a PID. So this problem as stated doesn't work. The question likely intended a different ring or ideal.

Problem 9. Show that if M is a finitely generated module over a Noetherian ring R, then every submodule of M is finitely generated.

Solution.
Solution.

Since M is finitely generated, there is a surjection $R^n \to M$ for some n. Let $N \subseteq M$ be a submodule. The preimage of N under this surjection is a submodule of R^n.

Since R is Noetherian, R^n is Noetherian (finite direct sums of Noetherian modules are Noetherian). Thus, every submodule of R^n is finitely generated. In particular, the preimage of N is finitely generated, say by $m_1, \ldots, m_k$. Then the images of $m_1, \ldots, m_k$ in M generate N.

Problem 10. Prove that $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} = 0$.

Solution.
Solution.

Since $\gcd(2, 3) = 1$, we have $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/1\mathbb{Z} = 0$.

Alternatively, every element is of the form $\overline{a} \otimes \overline{b}$. We have:

$$\overline{a} \otimes \overline{b} = \overline{a} \otimes (3 \cdot \overline{b}) = 3(\overline{a} \otimes \overline{b})$$

But also, $3\overline{a} = \overline{a}$ in $\mathbb{Z}/2\mathbb{Z}$ (since $3 \equiv 1 \pmod 2$), so:

$$3(\overline{a} \otimes \overline{b}) = (3\overline{a}) \otimes \overline{b} = \overline{a} \otimes \overline{b}$$

Wait, that's not quite right. Let me reconsider. We have $2\overline{a} = 0$ in $\mathbb{Z}/2\mathbb{Z}$ and $3\overline{b} = 0$ in $\mathbb{Z}/3\mathbb{Z}$. Thus:

$$2(\overline{a} \otimes \overline{b}) = (2\overline{a}) \otimes \overline{b} = 0 \otimes \overline{b} = 0$$

$$3(\overline{a} \otimes \overline{b}) = \overline{a} \otimes (3\overline{b}) = \overline{a} \otimes 0 = 0$$

Since $\gcd(2, 3) = 1$, there exist integers s, t with $2s + 3t = 1$. Then:

$$\overline{a} \otimes \overline{b} = 1 \cdot (\overline{a} \otimes \overline{b}) = (2s + 3t)(\overline{a} \otimes \overline{b}) = 2s(\overline{a} \otimes \overline{b}) + 3t(\overline{a} \otimes \overline{b}) = s \cdot 0 + t \cdot 0 = 0$$

Thus, every element is zero, so the tensor product is zero.

Problem 11. Show that a module M over a PID R is free if and only if it is projective.

Solution.
Solution.

A module P is **projective** if for every surjection $f : M \to N$ and every homomorphism $g : P \to N$, there exists a homomorphism $h : P \to M$ such that $f \circ h = g$ (i.e., every diagram can be "lifted").

(⇒) Free modules are projective: If P is free with basis B, then for any $g : P \to N$ and surjection $f : M \to N$, we can lift g by choosing preimages of $g(b)$ for each $b \in B$ and extending linearly.

(⇐) Over a PID, every projective module is free: This is a deep theorem. The proof uses the fact that every submodule of a free module over a PID is free, and that projective modules are direct summands of free modules.

Problem 12. Compute the invariant factors of the abelian group $\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/15\mathbb{Z}$.

Solution.
Solution.

The elementary divisors are the prime power factors of 6 and 15:

- $6 = 2 \cdot 3$, so elementary divisors from $\mathbb{Z}/6\mathbb{Z}$ are $2, 3$
- $15 = 3 \cdot 5$, so elementary divisors from $\mathbb{Z}/15\mathbb{Z}$ are $3, 5$

Collecting all elementary divisors: $2, 3, 3, 5$.

To find invariant factors, we group the elementary divisors so that each invariant factor divides the next. We pair up the largest prime powers:

- $d_1 = 2$
- $d_2 = 3 \cdot 3 \cdot 5 = 45$ (or we could do $d_1 = 3$, $d_2 = 30$, but we need $d_1 \mid d_2$)

Actually, let's be more careful. We have elementary divisors $2, 3, 3, 5$. To convert to invariant factors, we take the product of the largest power of each prime that appears:

- Largest power of 2: $2^1 = 2$
- Largest power of 3: $3^2 = 9$ (wait, we have two 3's, not 3^2)

Let me reconsider. The elementary divisors are $2, 3, 3, 5$. To find invariant factors, we compute:

- $d_k = \text{lcm of all elementary divisors} = \text{lcm}(2, 3, 3, 5) = 30$
- $d_{k-1} = \text{lcm of all but the smallest elementary divisor} = \text{lcm}(3, 3, 5) = 15$ (removing 2)

Wait, this isn't the right algorithm either. Let me use the correct method:

The invariant factors are obtained by taking products of elementary divisors in a specific way. For $2, 3, 3, 5$, we have:

- $d_1 = \gcd(6, 15) = 3$
- $d_2 = \text{lcm}(6, 15)/d_1 = 30/3 = 10$ (no, this isn't right either)

Actually, the correct answer is: $\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/15\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/30\mathbb{Z}$ (invariant factors 3, 30), since $3 \mid 30$ and $3 \cdot 30 = 90 = 6 \cdot 15$.

Problem 13. Prove that $M \otimes_R N \cong N \otimes_R M$ (commutativity of tensor product).

Solution.
Solution.

Define $\varphi : M \times N \to N \otimes_R M$ by $\varphi(m, n) = n \otimes m$. This is R-bilinear, so by the universal property, it induces a homomorphism $\widetilde{\varphi} : M \otimes_R N \to N \otimes_R M$ with $\widetilde{\varphi}(m \otimes n) = n \otimes m$.

Similarly, we get a homomorphism $\widetilde{\psi} : N \otimes_R M \to M \otimes_R N$ with $\widetilde{\psi}(n \otimes m) = m \otimes n$.

These are mutual inverses: $\widetilde{\psi}(\widetilde{\varphi}(m \otimes n)) = \widetilde{\psi}(n \otimes m) = m \otimes n$, and similarly for the other composition.

Problem 14. Show that if M is a finitely generated torsion-free module over a PID, then M is free.

Solution.
Solution.

By the structure theorem, $M \cong R^r \oplus R/(d_1) \oplus \cdots \oplus R/(d_k)$. Since M is torsion-free, the torsion part must be zero, i.e., $k = 0$. Thus, $M \cong R^r$, which is free.

Problem 15. Prove that $\mathrm{Hom}_R(M \oplus N, P) \cong \mathrm{Hom}_R(M, P) \oplus \mathrm{Hom}_R(N, P)$.

Solution.
Solution.

Define $\varphi : \mathrm{Hom}_R(M \oplus N, P) \to \mathrm{Hom}_R(M, P) \oplus \mathrm{Hom}_R(N, P)$ by $\varphi(f) = (f|_M, f|_N)$ (restriction to M and N).

Define $\psi : \mathrm{Hom}_R(M, P) \oplus \mathrm{Hom}_R(N, P) \to \mathrm{Hom}_R(M \oplus N, P)$ by $\psi(g, h)(m, n) = g(m) + h(n)$.

One verifies that φ and ψ are mutual inverses and are R-module homomorphisms.

Problem 16. Show that the sequence $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ does not split.

Solution.
Solution.

If the sequence split, then by the Splitting Lemma, $\mathbb{Z} \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. But the right side has torsion (the element $(0, \overline{1})$ satisfies $2(0, \overline{1}) = (0, \overline{0}) = 0$), while $\mathbb{Z}$ is torsion-free. This is a contradiction, so the sequence does not split.

Problem 17. Compute $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$.

Solution.
Solution.

Every element is of the form $\frac{a}{b} \otimes \overline{m}$ where $a, b \in \mathbb{Z}$, $b \neq 0$, and $\overline{m} \in \mathbb{Z}/n\mathbb{Z}$. We have:

$$\frac{a}{b} \otimes \overline{m} = \frac{1}{b} \otimes (a\overline{m}) = \frac{1}{b} \otimes \overline{am}$$

Also, $n\overline{m} = \overline{0}$, so:

$$\frac{a}{b} \otimes \overline{m} = \frac{a}{b} \otimes (n \cdot \frac{\overline{m}}{n})$$

Wait, this doesn't make sense. Let me reconsider. We have:

$$n\left(\frac{a}{b} \otimes \overline{m}\right) = \frac{a}{b} \otimes (n\overline{m}) = \frac{a}{b} \otimes \overline{0} = 0$$

So every element is killed by n. But $\mathbb{Q}$ is divisible, so for any $\frac{a}{b} \in \mathbb{Q}$, we have $\frac{a}{b} = n \cdot \frac{a}{nb}$. Thus:

$$\frac{a}{b} \otimes \overline{m} = n \cdot \frac{a}{nb} \otimes \overline{m} = \frac{a}{nb} \otimes (n\overline{m}) = \frac{a}{nb} \otimes \overline{0} = 0$$

Therefore, $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = 0$.

Problem 18. Prove that every short exact sequence of vector spaces splits.

> ***Solution.***
> Solution.
>
> Let $0 \to V \xrightarrow{f} W \xrightarrow{g} U \to 0$ be a short exact sequence of vector spaces over a field F. Since U is a vector space, it is free (every vector space has a basis). By Problem 4, the sequence splits.
>
> Alternatively, we can use the fact that $W \cong V \oplus U$ as vector spaces (by choosing a complement of $\ker(g) = \mathrm{Im}(f)$ in W).

Problem 19. Show that $\mathbb{Z}[x]$ is not a free $\mathbb{Z}$-module.

> ***Solution.***
> Solution.
>
> Suppose $\mathbb{Z}[x]$ were free with basis B. Since $\mathbb{Z}[x]$ is countably infinite, B is countable. Let $B = \{b_1, b_2, b_3, \ldots\}$.
>
> Each b_i is a polynomial, so b_i has some degree d_i. Let $d = \max\{d_1, d_2, \ldots, d_n\}$ for some large n. Then x^{d+1} cannot be in the span of $\{b_1, \ldots, b_n\}$ (since any linear combination has degree at most d). But x^{d+1} must be in the span of B, so it must involve b_i for $i > n$. Continuing this argument, we see that x^{d+1} requires infinitely many basis elements, contradicting the definition of a basis (which requires finite linear combinations).
>
> Therefore, $\mathbb{Z}[x]$ is not free as a $\mathbb{Z}$-module.

Problem 20. Prove that if R is a PID and M is a finitely generated R-module, then M has a finite free resolution (i.e., there exists an exact sequence $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ with each F_i free and n finite).

> ***Solution.***
> Solution.
>
> Since M is finitely generated, there is a surjection $F_0 \to M$ where F_0 is a finitely generated free module (i.e., $F_0 \cong R^n$). Let $K_0 = \ker(F_0 \to M)$. Then K_0 is a submodule of the free module F_0. Since R is a PID, K_0 is also free (submodules of free modules over PIDs are free).
>
> Thus, we have an exact sequence $0 \to K_0 \to F_0 \to M \to 0$. Since K_0 is free, we can stop here, giving a free resolution of length 1:
>
> $$0 \to K_0 \to F_0 \to M \to 0$$
>
> This shows that every finitely generated module over a PID has a finite free resolution of length at most 1. (In fact, the **projective dimension** of M is at most 1.)

❦

End of Chapter 21

Chapter 22: Category Theory

"Mathematics is the art of giving the same name to different things," wrote Henri Poincaré. Category theory takes this art to its zenith, revealing that seemingly disparate mathematical structures—groups, rings, topological spaces, vector spaces—are all instances of a single unifying concept: the category. In this chapter, we ascend to the highest level of abstraction in our journey through algebra, where we study not objects themselves but the *patterns of relationships* between objects.

Category theory, sometimes called "abstract nonsense" (affectionately, by its practitioners), provides a language for expressing mathematical ideas with unprecedented clarity and generality. It reveals deep connections between different areas of mathematics and provides powerful tools for proving theorems. As we shall see, many of the constructions we have encountered—products, quotients, free objects—are special cases of universal properties in category theory.

22.1 Categories and Functors

The Concept of a Category

A category consists of objects and morphisms (arrows) between objects, satisfying certain axioms. The key insight is that we focus not on the internal structure of objects but on the relationships between them.

> **Definition**
>
> Definition 22.1 (Category)
>
> A **category** $\mathcal{C}$ consists of:
>
> 1. A class of **objects**, denoted $\mathrm{Ob}(\mathcal{C})$
> 2. For each pair of objects A, B, a set of **morphisms** (or **arrows**) from A to B, denoted $\mathrm{Hom}_{\mathcal{C}}(A, B)$ or $\mathcal{C}(A, B)$
> 3. For each object A, an **identity morphism** $\mathrm{id}_A \in \mathcal{C}(A, A)$
> 4. A **composition operation**: for morphisms $f \in \mathcal{C}(A, B)$ and $g \in \mathcal{C}(B, C)$, a composite $g \circ f \in \mathcal{C}(A, C)$
>
> These must satisfy:
>
> - **Associativity:** $(h \circ g) \circ f = h \circ (g \circ f)$ whenever the compositions are defined
> - **Identity laws:** $f \circ \mathrm{id}_A = f$ and $\mathrm{id}_B \circ f = f$ for any $f \in \mathcal{C}(A, B)$

We often write $f : A \to B$ to denote a morphism from A to B. The objects are often called the "vertices" and the morphisms the "edges" of the category, though this geometric language can be misleading for large categories.

Example

Example 22.1 (The Category $\mathbf{Set}$)

The category $\mathbf{Set}$ has:

- Objects: all sets
- Morphisms: functions between sets
- Composition: usual function composition
- Identity: identity function $\mathrm{id}_X(x) = x$

This is the prototypical category and serves as a model for many others.

Example

Example 22.2 (The Category $\mathbf{Grp}$)

The category $\mathbf{Grp}$ has:

- Objects: all groups
- Morphisms: group homomorphisms
- Composition: composition of homomorphisms
- Identity: identity homomorphism

Similarly, we have categories $\mathbf{Ab}$ (abelian groups), $\mathbf{Ring}$ (rings), $\mathbf{Field}$ (fields), etc.

Example

Example 22.3 (The Category $\mathbf{Vect}_F$)

For a field F, the category $\mathbf{Vect}_F$ has:

- Objects: vector spaces over F
- Morphisms: linear transformations
- Composition: composition of linear maps
- Identity: identity linear transformation

Example

Example 22.4 (The Category $\mathbf{Mod}_R$)

For a ring R, the category $\mathbf{Mod}_R$ has:

- Objects: R-modules
- Morphisms: R-module homomorphisms
- Composition: composition of homomorphisms
- Identity: identity homomorphism

This generalizes $\mathbf{Vect}_F$ (taking $R = F$) and $\mathbf{Ab}$ (taking $R = \mathbb{Z}$).

Example

Example 22.5 (The Category $\mathbf{Top}$)

The category $\mathbf{Top}$ has:

- Objects: topological spaces
- Morphisms: continuous functions
- Composition: composition of continuous functions
- Identity: identity function (which is continuous)

This shows that category theory applies beyond algebra to topology and analysis.

Example

Example 22.6 (Posets as Categories)

A partially ordered set $(P, \leq)$ can be viewed as a category:

- Objects: elements of P
- Morphisms: there is a unique morphism $x \to y$ if and only if $x \leq y$
- Composition: transitivity of $\leq$
- Identity: reflexivity of $\leq$

This example shows that categories generalize order relations.

Definition

Definition 22.2 (Isomorphism)

A morphism $f : A \to B$ in a category $\mathcal{C}$ is an **isomorphism** if there exists a morphism $g : B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. We write $A \cong B$ if there exists an isomorphism between A and B.

In $\mathbf{Set}$, isomorphisms are bijections. In $\mathbf{Grp}$, they are group isomorphisms. In $\mathbf{Top}$, they are homeomorphisms. Category theory unifies these concepts.

Functors: Morphisms Between Categories

Just as morphisms relate objects within a category, functors relate categories themselves. A functor is a "structure-preserving map" between categories.

Definition

Definition 22.3 (Functor)

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **covariant functor** $F : \mathcal{C} \to \mathcal{D}$ consists of:

1. An **object function**: for each object $A \in \mathcal{C}$, an object $F(A) \in \mathcal{D}$
2. A **morphism function**: for each morphism $f : A \to B$ in $\mathcal{C}$, a morphism $F(f) : F(A) \to F(B)$ in $\mathcal{D}$

satisfying:

- $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ (preserves identities)
- $F(g \circ f) = F(g) \circ F(f)$ (preserves composition)

Definition

Definition 22.4 (Contravariant Functor)

A **contravariant functor** $F : \mathcal{C} \to \mathcal{D}$ is like a covariant functor except that it **reverses arrows**: for $f : A \to B$ in $\mathcal{C}$, we have $F(f) : F(B) \to F(A)$ in $\mathcal{D}$, and $F(g \circ f) = F(f) \circ F(g)$.

Example

Example 22.7 (Forgetful Functors)

The **forgetful functor** $U : \mathbf{Grp} \to \mathbf{Set}$ "forgets" the group structure:

- $U(G)$ is the underlying set of the group G
- $U(\varphi)$ is the underlying function of the homomorphism φ

Similarly, we have forgetful functors $\mathbf{Ring} \to \mathbf{Ab}$ (forget multiplication), $\mathbf{Top} \to \mathbf{Set}$ (forget topology), etc.

Example

Example 22.8 (Free Functor)

The **free group functor** $F : \mathbf{Set} \to \mathbf{Grp}$ assigns to each set X the free group $F(X)$ on X. For a function $f : X \to Y$, $F(f) : F(X) \to F(Y)$ is the induced homomorphism.

This is "left adjoint" to the forgetful functor (a concept we'll explore later).

Example

Example 22.9 (Hom Functors)

Fix an object A in a category $\mathcal{C}$. The **covariant Hom functor** $\mathrm{Hom}(A, -) : \mathcal{C} \to \mathbf{Set}$ is defined by:

- On objects: $B \mapsto \mathrm{Hom}(A, B)$
- On morphisms: for $g : B \to C$, the map $\mathrm{Hom}(A, g) : \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$ is given by $f \mapsto g \circ f$

Similarly, the **contravariant Hom functor** $\mathrm{Hom}(-, A) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is defined by:

- On objects: $B \mapsto \mathrm{Hom}(B, A)$
- On morphisms: for $g : B \to C$, the map $\mathrm{Hom}(g, A) : \mathrm{Hom}(C, A) \to \mathrm{Hom}(B, A)$ is given by $f \mapsto f \circ g$

Example

Example 22.10 (Fundamental Group Functor)

The fundamental group construction $\pi_1 : \mathbf{Top}_* \to \mathbf{Grp}$ is a functor from the category of pointed topological spaces to groups. It assigns to each pointed space (X, x_0) its fundamental group $\pi_1(X, x_0)$, and to each continuous map $f : (X, x_0) \to (Y, y_0)$ the induced homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$.

Properties of Functors

Proposition

Proposition 22.1

A functor $F : \mathcal{C} \to \mathcal{D}$ maps isomorphisms to isomorphisms. That is, if $f : A \to B$ is an isomorphism in $\mathcal{C}$, then $F(f) : F(A) \to F(B)$ is an isomorphism in $\mathcal{D}$.

Proof.

Proof.

If $f : A \to B$ is an isomorphism with inverse $g : B \to A$, then $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. Applying F, we get $F(g) \circ F(f) = F(g \circ f) = F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ and similarly $F(f) \circ F(g) = \mathrm{id}_{F(B)}$. Thus, $F(g)$ is the inverse of $F(f)$. □

∎

However, functors need not reflect isomorphisms: $F(f)$ being an isomorphism does not imply f is an isomorphism. For example, the forgetful functor $\mathbf{Grp} \to \mathbf{Set}$ sends the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ (not an isomorphism of groups) to an injection of sets, but this doesn't make the original map an isomorphism.

22.2 Natural Transformations

If functors are the morphisms between categories, then natural transformations are the "morphisms between functors." They capture the idea of a "natural" or "canonical" relationship between two functors.

Definition

Definition 22.5 (Natural Transformation)

Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A **natural transformation** $\eta : F \Rightarrow G$ is a family of morphisms $\{\eta_A : F(A) \to G(A)\}_{A \in \mathcal{C}}$ (one for each object A in $\mathcal{C}$) such that for every morphism $f : A \to B$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{\eta_A} & G(A) \\ \downarrow F(f) & & \downarrow G(f) \\ F(B) & \xrightarrow{\eta_B} & G(B) \end{array}$$

That is, $G(f) \circ \eta_A = \eta_B \circ F(f)$. The morphisms η_A are called the **components** of the natural transformation.

The commutativity condition is the essence of "naturality"—it says that η respects the structure preserved by the functors F and G.

Definition

Definition 22.6 (Natural Isomorphism)

A natural transformation $\eta : F \Rightarrow G$ is a **natural isomorphism** if each component $\eta_A : F(A) \to G(A)$ is an isomorphism in $\mathcal{D}$. We write $F \cong G$ and say F and G are **naturally isomorphic**.

Example

Example 22.11 (Double Dual of Vector Spaces)

Let V be a finite-dimensional vector space over a field F, and let $V^* = \mathrm{Hom}(V, F)$ be its dual space. There is a natural isomorphism $V \cong V^{**}$ (the double dual).

Define $\eta_V : V \to V^{**}$ by $\eta_V(v)(\varphi) = \varphi(v)$ for $v \in V$ and $\varphi \in V^*$. This is an isomorphism (for finite-dimensional V), and it is natural: for any linear map $f : V \to W$, the diagram

$$\begin{array}{ccc} V & \overset{\eta_V}{\to} & V^{**} \\ \downarrow f & & \downarrow f^{**} \\ W & \overset{\eta_W}{\to} & W^{**} \end{array}$$

commutes. This is the "canonical" isomorphism between V and V^{**}, in contrast to the isomorphism $V \cong V^*$, which requires choosing a basis.

Example

Example 22.12 (Determinant as a Natural Transformation)

Consider the functor $GL_n : \mathbf{Field} \to \mathbf{Grp}$ that sends a field F to the group $GL_n(F)$ of invertible $n \times n$ matrices over F. There is a natural transformation $\det : GL_n \Rightarrow (-)^*$ (where $(-)^*$ sends F to its multiplicative group F^*) given by the determinant map $\det : GL_n(F) \to F^*$.

For any field homomorphism $\varphi : F \to K$, the diagram

$$\begin{array}{ccc} GL_n(F) & \overset{\det}{\to} & F^* \\ \downarrow \varphi_* & & \downarrow \varphi \\ GL_n(K) & \overset{\det}{\to} & K^* \end{array}$$

commutes, showing that the determinant is "natural."

The Yoneda Lemma

One of the most profound results in category theory is the Yoneda Lemma, which says that an object is completely determined by the morphisms into (or out of) it.

Theorem

Theorem 22.1 (Yoneda Lemma)

Let $\mathcal{C}$ be a category, A an object in $\mathcal{C}$, and $F : \mathcal{C} \to \mathbf{Set}$ a functor. Then there is a bijection:

$$\mathrm{Nat}(\mathrm{Hom}(A, -), F) \cong F(A)$$

where $\mathrm{Nat}(\mathrm{Hom}(A, -), F)$ denotes the set of natural transformations from $\mathrm{Hom}(A, -)$ to F. Moreover, this bijection is natural in both A and F.

> ***Proof.***
> Proof (Sketch).
>
> Given a natural transformation $\eta : \mathrm{Hom}(A, -) \Rightarrow F$, we associate to it the element $\eta_A(\mathrm{id}_A) \in F(A)$. Conversely, given an element $x \in F(A)$, we define a natural transformation η^x by $\eta^x_B(f) = F(f)(x)$ for $f \in \mathrm{Hom}(A, B)$. One verifies that these constructions are mutual inverses and that η^x is indeed natural. □

∎

Corollary 22.1 (Yoneda Embedding)

The Yoneda embedding $\mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ given by $A \mapsto \mathrm{Hom}(-, A)$ is fully faithful. That is, $\mathrm{Hom}(A, B) \cong \mathrm{Nat}(\mathrm{Hom}(-, A), \mathrm{Hom}(-, B))$.

This says that an object A is completely determined (up to isomorphism) by the functor $\mathrm{Hom}(-, A)$. In other words, to understand an object, it suffices to understand all the morphisms into it.

Equivalence of Categories

> **Definition**
>
> Definition 22.7 (Equivalence of Categories)
>
> Two categories $\mathcal{C}$ and $\mathcal{D}$ are **equivalent** if there exist functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F \cong \mathrm{id}_{\mathcal{C}}$ and $F \circ G \cong \mathrm{id}_{\mathcal{D}}$ (natural isomorphisms).

Equivalence is weaker than isomorphism of categories (where we would require $G \circ F = \mathrm{id}_{\mathcal{C}}$ and $F \circ G = \mathrm{id}_{\mathcal{D}}$ on the nose). Equivalent categories are "the same for all practical purposes"—they have the same categorical properties.

> **Example**
>
> Example 22.13 (Finite-Dimensional Vector Spaces)
>
> The category of finite-dimensional vector spaces over F is equivalent to the category whose objects are natural numbers $\{0, 1, 2, 3, \ldots\}$ and whose morphisms $n \to m$ are $m \times n$ matrices over F. The equivalence is given by sending a vector space V to its dimension $\dim(V)$.

22.3 Universal Properties

Many constructions in algebra—products, coproducts, quotients, free objects—can be characterized by **universal properties**. Category theory provides a unified framework for understanding these constructions.

Products and Coproducts

Definition

Definition 22.8 (Product)

Let A and B be objects in a category $\mathcal{C}$. A **product** of A and B is an object $A \times B$ together with morphisms $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ (called **projections**) such that for any object C and morphisms $f : C \to A$ and $g : C \to B$, there exists a unique morphism $\langle f, g \rangle : C \to A \times B$ making the following diagram commute:

$$\begin{array}{ccccc} & & C & & \\ & & \downarrow \langle f,g \rangle & & \\ A & \xleftarrow{\pi_1} & A \times B & \xrightarrow{\pi_2} & B \\ & \nwarrow f & & \nearrow g & \end{array}$$

In $\mathbf{Set}$, the product is the Cartesian product. In $\mathbf{Grp}$, it's the direct product of groups. In $\mathbf{Top}$, it's the product topology. The universal property unifies all these constructions.

Definition

Definition 22.9 (Coproduct)

The **coproduct** (or **sum**) $A \sqcup B$ is the dual notion: it comes with morphisms $\iota_1 : A \to A \sqcup B$ and $\iota_2 : B \to A \sqcup B$ such that for any $f : A \to C$ and $g : B \to C$, there exists a unique $[f, g] : A \sqcup B \to C$ making the appropriate diagram commute.

In $\mathbf{Set}$, the coproduct is the disjoint union. In $\mathbf{Grp}$, it's the free product. In $\mathbf{Ab}$, it's the direct sum (which coincides with the product for finitely many objects).

Limits and Colimits

Products and coproducts are special cases of **limits** and **colimits**, which are among the most important concepts in category theory.

Definition

Definition 22.10 (Limit)

Let $D : \mathcal{J} \to \mathcal{C}$ be a functor (called a **diagram** of shape $\mathcal{J}$ in $\mathcal{C}$). A **limit** of D is an object $\lim D$ together with morphisms $\pi_J : \lim D \to D(J)$ for each $J \in \mathcal{J}$, satisfying a universal property: for any object C with morphisms $f_J : C \to D(J)$ compatible with the diagram, there exists a unique morphism $u : C \to \lim D$ such that $\pi_J \circ u = f_J$ for all J.

Examples of limits include:

- **Products:** limit of a discrete diagram

- **Equalizers:** limit of a parallel pair of morphisms
- **Pullbacks:** limit of a cospan
- **Inverse limits:** limit of a directed system

The dual notion is the **colimit**, which includes coproducts, coequalizers, pushouts, and direct limits.

Adjoint Functors

Adjoint functors are pairs of functors that stand in a special relationship, generalizing many familiar constructions.

> **Definition**
>
> Definition 22.11 (Adjoint Functors)
>
> Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors. We say F is **left adjoint** to G (and G is **right adjoint** to F), written $F \dashv G$, if there is a natural bijection:
>
> $$\mathrm{Hom}_{\mathcal{D}}(F(A), B) \cong \mathrm{Hom}_{\mathcal{C}}(A, G(B))$$
>
> for all objects $A \in \mathcal{C}$ and $B \in \mathcal{D}$.

> **Example**
>
> Example 22.14 (Free-Forgetful Adjunction)
>
> The free group functor $F : \mathbf{Set} \to \mathbf{Grp}$ is left adjoint to the forgetful functor $U : \mathbf{Grp} \to \mathbf{Set}$. The bijection is:
>
> $$\mathrm{Hom}_{\mathbf{Grp}}(F(X), G) \cong \mathrm{Hom}_{\mathbf{Set}}(X, U(G))$$
>
> This says that group homomorphisms from the free group $F(X)$ to G correspond bijectively to set functions from X to the underlying set of G.

> **Example**
>
> Example 22.15 (Tensor-Hom Adjunction)
>
> For a fixed R-module M, the functor $M \otimes_R - : \mathbf{Mod}_R \to \mathbf{Mod}_R$ is left adjoint to $\mathrm{Hom}_R(M, -) : \mathbf{Mod}_R \to \mathbf{Mod}_R$. The bijection is:
>
> $$\mathrm{Hom}_R(M \otimes_R N, P) \cong \mathrm{Hom}_R(N, \mathrm{Hom}_R(M, P))$$

Theorem

Theorem 22.2 (Adjoints Preserve Limits/Colimits)

Left adjoints preserve colimits, and right adjoints preserve limits.

This theorem has many consequences. For example, since the free group functor is a left adjoint, it preserves coproducts: $F(X \sqcup Y) \cong F(X) * F(Y)$ (free product).

22.4 Abelian Categories

Abelian categories are categories that behave like the category of abelian groups or modules. They provide the natural setting for homological algebra.

Definition

Definition 22.12 (Abelian Category)

A category $\mathcal{C}$ is **abelian** if:

1. $\mathcal{C}$ has a zero object (an object that is both initial and terminal)
2. Every pair of objects has a product and a coproduct
3. Every morphism has a kernel and a cokernel
4. Every monomorphism is the kernel of its cokernel, and every epimorphism is the cokernel of its kernel

Example

Example 22.16 (Examples of Abelian Categories)

- $\mathbf{Ab}$ (abelian groups)
- $\mathbf{Mod}_R$ (R-modules for any ring R)
- $\mathbf{Vect}_F$ (vector spaces over a field F)
- Sheaves of abelian groups on a topological space

In an abelian category, we can define exact sequences, and many results from module theory generalize. For example, the Snake Lemma and Five Lemma hold in any abelian category.

Definition

Definition 22.13 (Exact Sequence in an Abelian Category)

A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ in an abelian category is **exact at** B if $\operatorname{im}(f) = \ker(g)$.

The theory of abelian categories allows us to do homological algebra in great generality, beyond just modules.

22.5 Galois Connections

We conclude this chapter by revisiting Galois theory from a categorical perspective. The Galois correspondence can be understood as a special case of a more general phenomenon.

> **Definition**
>
> Definition 22.14 (Galois Connection)
>
> Let $(P, \leq)$ and $(Q, \leq)$ be partially ordered sets. A **Galois connection** between P and Q is a pair of order-reversing maps $f : P \to Q$ and $g : Q \to P$ such that:
>
> $$p \leq g(q) \iff q \leq f(p)$$
>
> for all $p \in P$ and $q \in Q$.

Equivalently, f and g satisfy $p \leq g(f(p))$ and $q \leq f(g(q))$ for all p and q.

> **Example**
>
> Example 22.17 (Fundamental Theorem of Galois Theory)
>
> Let K/F be a Galois extension with Galois group G. The Galois correspondence establishes a Galois connection between:
>
> - P: the lattice of intermediate fields $F \subseteq E \subseteq K$, ordered by inclusion
> - Q: the lattice of subgroups of G, ordered by reverse inclusion
>
> The maps are $E \mapsto \mathrm{Gal}(K/E)$ and $H \mapsto K^H$ (the fixed field). These are order-reversing and satisfy the Galois connection property.

Galois connections appear throughout mathematics: in logic (syntax-semantics duality), in topology (closure operators), in algebra (ideals and varieties), and in many other areas. They capture the idea of a "duality" or "correspondence" between two ordered structures.

❦

> **Greek Meditation: Categories**
> **The Logos of Mathematics—Universal Patterns in Diversity**

The ancient Greeks sought the *logos*—the underlying rational principle that governs the cosmos. Heraclitus spoke of the logos as the unity behind apparent diversity, the pattern that connects all things. Category theory is the mathematical realization of this vision.

In category theory, we do not study individual objects in isolation but rather the *relationships* between objects. A group is not defined by its elements but by its homomorphisms to and from other groups. A topological space is characterized by the continuous maps it admits. This is the Platonic insight: the essence of a thing lies not in its material instantiation but in its Form—its place in the web of relationships.

The Yoneda Lemma embodies this philosophy: an object is completely determined by the morphisms into it. To know a thing is to know how it relates to all other things. This is reminiscent of Leibniz's principle of the identity of indiscernibles: two things are the same if and only if they have all the same properties. In category theory, "properties" are morphisms.

Natural transformations capture the idea of *canonical* or *natural* constructions—those that do not depend on arbitrary choices. The Greeks distinguished between *physis* (nature) and *nomos* (convention). A natural transformation is *physis*: it arises from the structure itself, not from our arbitrary choices. The isomorphism $V \cong V^{**}$ is natural; the isomorphism $V \cong V^*$ is not (it requires choosing a basis, a *nomos*).

Adjoint functors reveal deep dualities in mathematics. The free-forgetful adjunction shows that "freedom" and "forgetting" are dual concepts, two sides of the same coin. This echoes the Greek concept of *enantiodromia*—the tendency of things to transform into their opposites. Construction and deconstruction, synthesis and analysis, are adjoint processes.

And in abelian categories, we see the Pythagorean dream of unity: diverse mathematical structures—abelian groups, vector spaces, sheaves—are all instances of a single abstract concept. The theorems we prove in one context (the Snake Lemma, the Five Lemma) apply universally. This is the power of abstraction: to see the One in the Many, the universal in the particular.

Category theory is the culmination of the Greek philosophical project: to find the rational structure underlying reality, to discern the patterns that connect, to ascend from the particular to the universal. It is mathematics at its most philosophical, philosophy at its most mathematical.

Zen Meditation: The Way of Functors
Preserving Structure, Revealing Essence

In Zen, we speak of *suchness* (tathātā)—the true nature of things as they are, beyond concepts and categories. Yet category theory, paradoxically, reveals suchness through abstraction. By stripping away the inessential, we see what truly matters.

A functor preserves structure. It does not care about the "content" of objects—whether they are sets, groups, or topological spaces—but only about the *relationships* between them. This is the Zen teaching: do not cling to form, but attend to function. The essence of a thing is not what it is made of but what it does, how it relates.

Consider the forgetful functor: it "forgets" structure, revealing the underlying simplicity. A group becomes a set; a topological space becomes a set. This is like the Zen practice of *shikantaza* (just

sitting): we let go of all concepts, all structures, and return to bare awareness. What remains when we forget everything? The pure, unadorned reality.

But there is also the free functor, which *creates* structure from nothing. Given a set, we freely generate a group. This is the complementary practice: from emptiness, form arises. The free functor and the forgetful functor are adjoint—they are two aspects of the same process, like breathing in and breathing out.

Natural transformations teach us about *spontaneity*. A natural transformation arises without effort, without choice. It is not constructed but discovered. This is *wu wei* (non-action): the action that is not forced, the transformation that happens of itself. The double dual isomorphism $V \to V^{**}$ is natural—it requires no choices, no effort. It simply *is*.

The Yoneda Lemma is a koan: "An object is the collection of all morphisms into it." What is a thing? It is all the ways it can be approached, all the perspectives from which it can be seen. There is no "thing-in-itself," only relationships. This is the Buddhist teaching of *pratītyasamutpāda* (dependent origination): nothing exists independently; everything arises in dependence on everything else.

And in the study of limits and colimits, we see the interplay of convergence and divergence, unity and multiplicity. A limit brings many things together into one; a colimit allows one thing to become many. This is the dance of form and emptiness, the play of manifestation and dissolution.

Category theory, for all its abstraction, points us back to the concrete. By understanding the universal patterns, we see each particular instance more clearly. By ascending to the general, we return to the specific with new eyes. This is the Zen circle: we begin with mountains and rivers, we ascend to abstraction where mountains are not mountains and rivers are not rivers, and we return to mountains and rivers, seeing them as they truly are.

The Way of the Functor is the way of seeing essence through structure, of finding simplicity in complexity, of recognizing that all things are connected in the vast web of relationships that is mathematics—and reality itself.

❦

Practice Problems

Problem 1. Show that in the category $\mathbf{Set}$, every epimorphism is surjective and every monomorphism is injective.

> ***Solution.***
> Solution.
>
> **Monomorphism ⇒ Injective:** Let $f : A \to B$ be a monomorphism in $\mathbf{Set}$. Suppose $f(a_1) = f(a_2)$. Define $g_1, g_2 : \{*\} \to A$ by $g_1(*) = a_1$ and $g_2(*) = a_2$. Then $f \circ g_1 = f \circ g_2$ (both send $*$ to $f(a_1) = f(a_2)$). Since f is a monomorphism, $g_1 = g_2$, so $a_1 = a_2$. Thus, f is injective.

Epimorphism ⇒ Surjective: Let $f : A \to B$ be an epimorphism. Suppose f is not surjective, so there exists $b \in B$ not in the image of f. Let $C = \{0, 1\}$ and define $g_1, g_2 : B \to C$ by $g_1(x) = 0$ for all x, and $g_2(x) = 0$ if $x \neq b$, $g_2(b) = 1$. Then $g_1 \circ f = g_2 \circ f$ (since b is not in the image of f), but $g_1 \neq g_2$, contradicting that f is an epimorphism.

Problem 2. Prove that the composition of two functors is a functor.

Solution.
Solution.

Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors. Define $G \circ F : \mathcal{C} \to \mathcal{E}$ by $(G \circ F)(A) = G(F(A))$ on objects and $(G \circ F)(f) = G(F(f))$ on morphisms.

Preserves identities: $(G \circ F)(\mathrm{id}_A) = G(F(\mathrm{id}_A)) = G(\mathrm{id}_{F(A)}) = \mathrm{id}_{G(F(A))} = \mathrm{id}_{(G \circ F)(A)}$.

Preserves composition: $(G \circ F)(g \circ f) = G(F(g \circ f)) = G(F(g) \circ F(f)) = G(F(g)) \circ G(F(f)) = (G \circ F)(g) \circ (G \circ F)(f)$.

Problem 3. Show that the product in a category, if it exists, is unique up to unique isomorphism.

Solution.
Solution.

Suppose (P, π_1, π_2) and (Q, ρ_1, ρ_2) are both products of A and B. By the universal property of P, there exists a unique morphism $f : P \to Q$ such that $\rho_1 \circ f = \pi_1$ and $\rho_2 \circ f = \pi_2$. Similarly, by the universal property of Q, there exists a unique $g : Q \to P$ such that $\pi_1 \circ g = \rho_1$ and $\pi_2 \circ g = \rho_2$.

Consider $g \circ f : P \to P$. We have $\pi_1 \circ (g \circ f) = (\pi_1 \circ g) \circ f = \rho_1 \circ f = \pi_1$ and similarly $\pi_2 \circ (g \circ f) = \pi_2$. But id_P also satisfies $\pi_1 \circ \mathrm{id}_P = \pi_1$ and $\pi_2 \circ \mathrm{id}_P = \pi_2$. By uniqueness in the universal property, $g \circ f = \mathrm{id}_P$. Similarly, $f \circ g = \mathrm{id}_Q$. Thus, f and g are mutual inverses, so $P \cong Q$.

Problem 4. Verify that the determinant map $\det : GL_n(F) \to F^*$ is a natural transformation (as in Example 22.12).

Solution.
Solution.

We need to show that for any field homomorphism $\varphi : F \to K$, the diagram commutes. Let $A \in GL_n(F)$. Then $\varphi_*(A)$ is the matrix obtained by applying φ to each entry of A. We have:

$$\det(\varphi_*(A)) = \det((\varphi(a_{ij}))) = \varphi(\det((a_{ij}))) = \varphi(\det(A))$$

where the second equality uses the fact that the determinant formula involves only addition and multiplication, which φ preserves. Thus, $\det \circ \varphi_* = \varphi \circ \det$, so the diagram commutes.

Problem 5. Prove that if $F \dashv G$ (F is left adjoint to G), then F preserves colimits and G preserves limits.

Solution.

Solution (Sketch).

We prove that G preserves limits; the proof for F and colimits is dual. Let $D : \mathcal{J} \to \mathcal{D}$ be a diagram with limit $L = \lim D$ in $\mathcal{D}$. We want to show that $G(L)$ is the limit of $G \circ D$ in $\mathcal{C}$.

For any object $C \in \mathcal{C}$, we have:

$$\mathrm{Hom}(C, G(L)) \cong \mathrm{Hom}(F(C), L) \cong \lim \mathrm{Hom}(F(C), D(-)) \cong \lim \mathrm{Hom}(C, G(D(-))) \cong$$

where the first isomorphism is the adjunction, the second is the definition of limit, the third is the adjunction again, and the fourth is the definition of limit. By Yoneda, $G(L) \cong \lim G \circ D$.

Problem 6. Show that in an abelian category, every morphism can be factored as an epimorphism followed by a monomorphism.

Solution.
Solution.

Let $f : A \to B$ be a morphism in an abelian category. Define $I = \mathrm{im}(f) = \ker(\mathrm{coker}(f))$. Then we have a factorization $A \to I \to B$ where $A \to I$ is the cokernel of $\ker(f)$ (hence an epimorphism) and $I \to B$ is the kernel of $\mathrm{coker}(f)$ (hence a monomorphism). This is the **image factorization** of f.

Problem 7. Show that the Yoneda embedding $\mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is fully faithful.

Solution.
Solution.

We need to show that $\mathrm{Hom}_{\mathcal{C}}(A, B) \cong \mathrm{Nat}(\mathrm{Hom}(-, A), \mathrm{Hom}(-, B))$. By the Yoneda Lemma, $\mathrm{Nat}(\mathrm{Hom}(-, A), \mathrm{Hom}(-, B)) \cong \mathrm{Hom}(-, B)(A) = \mathrm{Hom}(A, B)$. This bijection is natural and is given by sending $f : A \to B$ to the natural transformation η^f with components $\eta^f_C(g) = f \circ g$ for $g \in \mathrm{Hom}(C, A)$.

Problem 8. Show that in $\mathbf{Grp}$, the coproduct of two groups is their free product.

Solution.
Solution.

Let G and H be groups. The free product $G * H$ consists of reduced words in elements of G and H, with multiplication by concatenation and reduction. There are natural inclusions $\iota_G : G \to G * H$ and $\iota_H : H \to G * H$.

To verify the universal property, let $\varphi_G : G \to K$ and $\varphi_H : H \to K$ be group homomorphisms. Define $\psi : G * H \to K$ by $\psi(g_1 h_1 g_2 h_2 \cdots) = \varphi_G(g_1)\varphi_H(h_1)\varphi_G(g_2)\varphi_H(h_2)\cdots$. This is well-defined (independent of the choice of reduced word) and is a homomorphism. Moreover, $\psi \circ \iota_G = \varphi_G$ and $\psi \circ \iota_H = \varphi_H$, and ψ is the unique such homomorphism. Thus, $G * H$ is the coproduct.

Problem 9. Show that a functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if it is fully faithful and essentially surjective (every object of $\mathcal{D}$ is isomorphic to $F(A)$ for some $A \in \mathcal{C}$).

Solution.

Solution (Sketch).

(⇒) If F is an equivalence with quasi-inverse G, then F is fully faithful (since $\mathrm{Hom}(A,B) \cong \mathrm{Hom}(GF(A), GF(B)) \cong \mathrm{Hom}(F(A), F(B))$) and essentially surjective (since $FG(D) \cong D$ for all $D \in \mathcal{D}$).

(⇐) If F is fully faithful and essentially surjective, construct $G : \mathcal{D} \to \mathcal{C}$ by choosing for each $D \in \mathcal{D}$ an object $G(D) \in \mathcal{C}$ with $F(G(D)) \cong D$. Extend to morphisms using full faithfulness. One verifies that G is a quasi-inverse to F.

Problem 10. Show that the category of finite sets and bijections is equivalent to the category whose objects are natural numbers and whose morphisms are bijections between sets of those sizes.

Solution.

Solution.

Define $F : \mathbf{FinSet}_{\cong} \to \mathbf{Nat}_{\cong}$ by $F(X) = |X|$ (the cardinality of X). For a bijection $f : X \to Y$, $F(f)$ is the bijection $\{1, \dots, |X|\} \to \{1, \dots, |Y|\}$ obtained by choosing bijections $X \cong \{1, \dots, |X|\}$ and $Y \cong \{1, \dots, |Y|\}$ and composing.

This functor is essentially surjective (every natural number is the cardinality of some finite set) and fully faithful (bijections between sets of the same size correspond to bijections between the corresponding natural numbers). Thus, it is an equivalence.

Problem 11. Verify that the tensor product functor $M \otimes_R - : \mathbf{Mod}_R \to \mathbf{Mod}_R$ is right exact.

Solution.

Solution.

Let $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence of R-modules. We need to show that $M \otimes A \xrightarrow{\mathrm{id} \otimes f} M \otimes B \xrightarrow{\mathrm{id} \otimes g} M \otimes C \to 0$ is exact.

Exactness at $M \otimes C$: Since g is surjective, $\mathrm{id} \otimes g$ is surjective (tensor product preserves surjections).

Exactness at $M \otimes B$: We have $(\mathrm{id} \otimes g) \circ (\mathrm{id} \otimes f) = \mathrm{id} \otimes (g \circ f) = \mathrm{id} \otimes 0 = 0$, so $\mathrm{im}(\mathrm{id} \otimes f) \subseteq \ker(\mathrm{id} \otimes g)$. For the reverse inclusion, one uses the fact that $M \otimes C \cong (M \otimes B)/(M \otimes \ker(g))$ and $\ker(g) = \mathrm{im}(f)$, so $\ker(\mathrm{id} \otimes g) = M \otimes \mathrm{im}(f) = \mathrm{im}(\mathrm{id} \otimes f)$.

Problem 12. Prove that in the category $\mathbf{Top}$, not every epimorphism is surjective. (Hint: Consider the inclusion $\mathbb{Q} \hookrightarrow \mathbb{R}$ with the usual topology.)

Solution.

Solution.

The inclusion $i : \mathbb{Q} \to \mathbb{R}$ is an epimorphism in $\mathbf{Top}$ but not surjective. To see that it's an epimorphism, suppose $f, g : \mathbb{R} \to X$ are continuous maps with $f \circ i = g \circ i$. Then f and g agree on the dense subset $\mathbb{Q}$. If X is Hausdorff, then $f = g$ (continuous functions that agree on

a dense subset of a Hausdorff space are equal). Thus, i is an epimorphism in the category of Hausdorff spaces, and hence in $\mathbf{Top}$.

Problem 13. Show that the forgetful functor $U : \mathbf{Grp} \to \mathbf{Set}$ has a left adjoint (the free group functor) and a right adjoint (the functor sending a set X to the group of all functions $X \to \mathbb{Z}$ under pointwise addition).

Solution.
Solution.

Left adjoint: The free group functor $F : \mathbf{Set} \to \mathbf{Grp}$ satisfies $\mathrm{Hom}_{\mathbf{Grp}}(F(X), G) \cong \mathrm{Hom}_{\mathbf{Set}}(X, U(G))$, as shown in Example 22.14.

Right adjoint: Define $R : \mathbf{Set} \to \mathbf{Grp}$ by $R(X) = \mathbb{Z}^X$ (functions $X \to \mathbb{Z}$ with pointwise addition). Then $\mathrm{Hom}_{\mathbf{Set}}(U(G), X) \cong \mathrm{Hom}_{\mathbf{Grp}}(G, R(X))$: a set function $G \to X$ corresponds to a group homomorphism $G \to \mathbb{Z}^X$ by sending $g \mapsto (x \mapsto \delta_{f(g),x})$ (where δ is the Kronecker delta). Wait, this doesn't quite work. Let me reconsider.

Actually, the right adjoint is $R(X) = \mathrm{Hom}_{\mathbf{Set}}(X, \mathbb{Z})$ where $\mathbb{Z}$ has the discrete topology. Then $\mathrm{Hom}_{\mathbf{Set}}(U(G), X) \cong \mathrm{Hom}_{\mathbf{Grp}}(G, \mathbb{Z}^X)$ by the adjunction between products and exponentials in $\mathbf{Set}$ and $\mathbf{Grp}$.

Problem 14. Prove that the category $\mathbf{Mod}_R$ is abelian.

Solution.
Solution.

We verify the axioms:

1. **Zero object:** The zero module $\{0\}$ is both initial and terminal.
2. **Products and coproducts:** The direct sum $M \oplus N$ serves as both product and coproduct (for finitely many modules).
3. **Kernels and cokernels:** For a homomorphism $f : M \to N$, $\ker(f) = \{m \in M : f(m) = 0\}$ and $\mathrm{coker}(f) = N/\mathrm{im}(f)$.
4. **Monomorphisms and epimorphisms:** Every monomorphism is the kernel of its cokernel, and every epimorphism is the cokernel of its kernel (this follows from the First Isomorphism Theorem).

Problem 15. Show that a Galois connection between posets $(P, \leq)$ and $(Q, \leq)$ induces an order-preserving bijection between the set of fixed points $\{p \in P : p = g(f(p))\}$ and $\{q \in Q : q = f(g(q))\}$.

Solution.
Solution.

Let $P' = \{p \in P : p = g(f(p))\}$ and $Q' = \{q \in Q : q = f(g(q))\}$. Define $\varphi : P' \to Q'$ by $\varphi(p) = f(p)$ and $\psi : Q' \to P'$ by $\psi(q) = g(q)$.

Well-defined: If $p \in P'$, then $f(g(f(p))) = f(g(f(g(f(p))))) \leq f(p)$ (using $q \leq f(g(q))$ and monotonicity). But also $f(p) \leq f(g(f(p)))$ (using $p \leq g(f(p)) = p$ and monotonicity). Thus, $f(p) = f(g(f(p)))$, so $f(p) \in Q'$. Similarly, ψ is well-defined.

Inverses: For $p \in P'$, $\psi(\varphi(p)) = g(f(p)) = p$. For $q \in Q'$, $\varphi(\psi(q)) = f(g(q)) = q$. Thus, φ and ψ are mutual inverses.

Order-preserving: If $p_1 \leq p_2$ in P', then $f(p_1) \geq f(p_2)$ (since f is order-reversing). Wait, this gives the wrong direction. Let me reconsider the setup.

Actually, the bijection is order-reversing, not order-preserving, because f and g are order-reversing. The correct statement is that $\varphi : P' \to Q'$ is an order-reversing bijection.

Problem 16. Prove that the category $\mathbf{Set}$ has all limits and colimits.

Solution.
Solution (Sketch).

$\mathbf{Set}$ is **complete** (has all limits) and **cocomplete** (has all colimits). For limits, the limit of a diagram $D : \mathcal{J} \to \mathbf{Set}$ is the subset of the product $\prod_{J \in \mathcal{J}} D(J)$ consisting of tuples (x_J) that are "compatible" with the morphisms in the diagram. For colimits, the colimit is constructed as a quotient of the disjoint union $\bigsqcup_{J \in \mathcal{J}} D(J)$ by an equivalence relation generated by the morphisms in the diagram.

Problem 17. Show that the fundamental group functor $\pi_1 : \mathbf{Top}_* \to \mathbf{Grp}$ does not preserve products.

Solution.
Solution.

The product in $\mathbf{Top}_*$ is the Cartesian product with the product topology and the product basepoint. We have $\pi_1((X, x_0) \times (Y, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$ (the fundamental group of a product is the product of the fundamental groups). So actually, π_1 *does* preserve products!

However, π_1 does not preserve all limits. For example, it does not preserve equalizers in general.

Problem 18. Prove that if $\mathcal{C}$ is a small category (the objects form a set, not a proper class), then the category $[\mathcal{C}, \mathbf{Set}]$ of functors $\mathcal{C} \to \mathbf{Set}$ is complete and cocomplete.

Solution.
Solution (Sketch).

Limits and colimits in $[\mathcal{C}, \mathbf{Set}]$ are computed "pointwise." If $D : \mathcal{J} \to [\mathcal{C}, \mathbf{Set}]$ is a diagram of functors, then $(\lim D)(C) = \lim(D(-)(C))$ (the limit in $\mathbf{Set}$ of the sets $D(J)(C)$). Since $\mathbf{Set}$ is complete and cocomplete, so is $[\mathcal{C}, \mathbf{Set}]$.

Problem 19. Show that the category of fields is not abelian.

Solution.
Solution.

The category $\mathbf{Field}$ does not have a zero object. There is no field that is both initial and terminal. The initial object is $\mathbb{Q}$ (or $\mathbb{F}_p$ in characteristic p), and there is no terminal object (for any field F, there are multiple field homomorphisms from F to various extension fields). Thus, $\mathbf{Field}$ is not abelian.

Problem 20. Prove that the Galois correspondence for a Galois extension K/F is a Galois connection (in the sense of Definition 22.14).

> ***Solution.***
> Solution.
>
> Let P be the lattice of intermediate fields $F \subseteq E \subseteq K$, ordered by inclusion, and let Q be the lattice of subgroups of $G = \mathrm{Gal}(K/F)$, ordered by reverse inclusion (i.e., $H_1 \leq_Q H_2$ iff $H_1 \supseteq H_2$). Define $f : P \to Q$ by $f(E) = \mathrm{Gal}(K/E)$ and $g : Q \to P$ by $g(H) = K^H$.
>
> These are order-reversing: if $E_1 \subseteq E_2$, then $\mathrm{Gal}(K/E_1) \supseteq \mathrm{Gal}(K/E_2)$, so $f(E_1) \geq_Q f(E_2)$. Similarly for g.
>
> The Galois connection property $E \subseteq g(f(E))$ and $H \subseteq f(g(H))$ (in the Q-order) follows from the Fundamental Theorem of Galois Theory: $E \subseteq K^{\mathrm{Gal}(K/E)}$ and $H \subseteq \mathrm{Gal}(K/K^H)$, with equality when K/F is Galois.

End of Chapter 22

Chapter 23: Representation Theory

"Symmetry is one idea by which man through the ages
has tried to comprehend and create order, beauty, and perfection."
— Hermann Weyl

In which we discover how abstract groups
manifest as matrices and linear transformations,
revealing the hidden symmetries of nature itself.

Chapter Overview:

- **23.1** — Group Representations
- **23.2** — Maschke's Theorem
- **23.3** — Character Theory
- **23.4** — Applications to Physics and Chemistry
- **23.5** — Representations of Lie Algebras

Chapter 23 — Representation Theory

Introduction: The Power of Representation

Representation theory stands as one of the most powerful and unifying theories in modern mathematics. It provides the crucial bridge between abstract algebra and linear algebra, allowing us to study abstract groups through their actions on vector spaces—a concrete setting where we have powerful computational tools at our disposal.

The fundamental insight is deceptively simple: to understand an abstract group G, we study how it can act linearly on vector spaces. Each such action gives us a *representation*—a homomorphism from G to the group of invertible matrices $\mathrm{GL}(V)$. Through this lens, abstract group elements become concrete matrices, and abstract group operations become matrix multiplication.

But representation theory is far more than a computational convenience. It reveals deep structural properties of groups, provides powerful invariants (characters) that distinguish groups, and has profound applications throughout mathematics, physics, and chemistry. The symmetries of molecules, the behavior of subatomic particles, and the vibrations of crystals all find their natural expression in the language of representation theory.

> ⚡ **Greek Meditation** ⚡

> **Representations: The Many Faces of One**
> Heraclitus taught that unity manifests in multiplicity—
> "The road up and the road down are one and the same."
> So too does an abstract group reveal its nature
> through infinite concrete realizations,
> each representation a different face of the same eternal Form.

The theory we develop in this chapter has its roots in the late 19th century with the work of Frobenius, but reached full maturity in the 20th century through contributions from Schur, Weyl, Brauer, and many others. Today it remains an active area of research with connections to number theory, algebraic geometry, and theoretical physics.

We begin with the fundamental definitions and examples, building intuition through explicit computations. We then prove Maschke's theorem, which guarantees that representations of finite groups over $\mathbb{C}$ are "semisimple"—they decompose completely into irreducible pieces. The theory of characters provides powerful tools for computing these decompositions and extracting group-theoretic information. Finally, we explore applications to physics and chemistry, and touch on the representation theory of Lie algebras.

23.1 — Group Representations

> **Definition**
>
> **Definition 23.1 (Group Representation).** Let G be a group and V a vector space over a field F. A *representation* of G on V is a group homomorphism
>
> $$\rho : G \to \mathrm{GL}(V)$$
>
> where $\mathrm{GL}(V)$ denotes the group of invertible linear transformations of V. The *degree* (or *dimension*) of the representation is $\dim(V)$.

Explicitly, a representation satisfies:

- $\rho(e) = I_V$, the identity transformation
- $\rho(g_1 g_2) = \rho(g_1) \circ \rho(g_2)$ for all $g_1, g_2 \in G$
- $\rho(g^{-1}) = \rho(g)^{-1}$ for all $g \in G$

When V is finite-dimensional with $\dim(V) = n$, choosing a basis transforms $\mathrm{GL}(V)$ into $\mathrm{GL}_n(F)$, the group of $n \times n$ invertible matrices. In this case, ρ assigns to each group element g an invertible matrix $\rho(g)$, and we call it a *matrix representation*.

Example

Example 23.1 (Trivial Representation). For any group G and any vector space V, define

$$\rho_{\text{triv}}(g) = I_V \quad \text{for all } g \in G.$$

This is the *trivial representation*. Every group element acts as the identity. While seemingly uninteresting, the trivial representation plays an important role in the theory.

Example

Example 23.2 (Sign Representation of S_n). Consider the symmetric group S_n and the one-dimensional vector space $V = F$. Define

$$\rho_{\text{sgn}}(\sigma) = \text{sgn}(\sigma) \cdot I_V$$

where $\text{sgn}(\sigma) \in \{1, -1\}$ is the sign of the permutation. As a matrix representation, $\rho_{\text{sgn}}(\sigma)$ is the 1×1 matrix $[\text{sgn}(\sigma)]$.

For example, in S_3:

$$\begin{aligned} \rho_{\text{sgn}}(e) &= [1] \\ \rho_{\text{sgn}}((12)) &= [-1] \\ \rho_{\text{sgn}}((123)) &= [1] \end{aligned}$$

Example

Example 23.3 (Regular Representation). Let G be a finite group with $|G| = n$. The *regular representation* is defined on the vector space $V = F[G]$ with basis $\{e_g : g \in G\}$ indexed by group elements. For $h \in G$, define

$$\rho_{\text{reg}}(h)(e_g) = e_{hg}.$$

This is a representation of degree $n = |G|$. The group acts on itself by left multiplication, and we view this action linearly.

The matrix entries are given by:

$$[\rho_{\text{reg}}(h)]_{g_1,g_2} = \begin{cases} 1 & \text{if } hg_2 = g_1 \\ 0 & \text{otherwise} \end{cases}$$

Each matrix $\rho_{\text{reg}}(h)$ is a permutation matrix with exactly one 1 in each row and column.

Example

Example 23.4 (Cyclic Group C_n). Let $G = C_n = \langle g \mid g^n = e \rangle$ be the cyclic group of order n. Since G is generated by g, a representation $\rho : C_n \to \mathrm{GL}_1(\mathbb{C})$ is determined by where we send g, provided $\rho(g)^n = 1$.

Thus we need a complex number λ with $\lambda^n = 1$—an nth root of unity. There are exactly n such roots:

$$\omega_k = e^{2\pi i k/n}, \quad k = 0, 1, \ldots, n-1.$$

Each gives a one-dimensional representation $\rho_k : C_n \to \mathbb{C}^*$ defined by

$$\rho_k(g^j) = \omega_k^j = e^{2\pi i j k/n}.$$

These are all the irreducible representations of C_n over $\mathbb{C}$.

Example

Example 23.5 (Dihedral Group D_n). Recall $D_n = \langle r, s \mid r^n = s^2 = e, srs = r^{-1} \rangle$ is the symmetry group of a regular n-gon. Consider the natural geometric representation on $\mathbb{R}^2$.

Place the n-gon centered at the origin. The rotation r rotates by $2\pi/n$, and the reflection s reflects across the x-axis. In matrix form:

$$\rho(r) = \begin{pmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{pmatrix}, \quad \rho(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This gives a faithful 2-dimensional representation of D_n.

Example

Example 23.6 (Complete Analysis of S_3). The symmetric group S_3 has order 6 and three conjugacy classes:

$$C_1 = \{e\}, \quad C_2 = \{(12), (13), (23)\}, \quad C_3 = \{(123), (132)\}.$$

There are three irreducible representations over $\mathbb{C}$:

(1) Trivial representation ρ_{triv}: degree 1, $\rho(g) = 1$ for all g.

(2) Sign representation ρ_{sgn}: degree 1, $\rho(\sigma) = \text{sgn}(\sigma)$.

(3) Standard representation ρ_{std}: degree 2. Consider $\mathbb{C}^3$ with basis $\{e_1, e_2, e_3\}$ and the subspace

$$V = \{(z_1, z_2, z_3) : z_1 + z_2 + z_3 = 0\}.$$

This is a 2-dimensional subspace. Let S_3 act by permuting coordinates:

$$\sigma \cdot (z_1, z_2, z_3) = (z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}).$$

Since σ preserves sums, V is invariant. Using the basis $\{e_1 - e_2, e_1 - e_3\}$ for V, we get:

$$\rho((12)) = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$$
$$\rho((123)) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

This is irreducible and faithful (different elements give different matrices).

These three representations exhaust the irreducibles of S_3. Note that $1^2 + 1^2 + 2^2 = 6 = |S_3|$, a relationship we'll explain later.

Equivalent Representations

Definition

Definition 23.2 (Intertwining Operator, Equivalence). Let $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ be representations. A linear map $\phi : V \to W$ is called an *intertwining operator* (or *homomorphism of representations*) if

$$\phi \circ \rho(g) = \sigma(g) \circ \phi \quad \text{for all } g \in G.$$

The representations ρ and σ are *equivalent* (or *isomorphic*) if there exists an invertible intertwining operator $\phi : V \to W$.

In matrix terms, if we have matrix representations $\rho(g)$ and $\sigma(g)$ (with respect to chosen bases), equivalence means there exists an invertible matrix P such that

$$\sigma(g) = P\rho(g)P^{-1} \quad \text{for all } g \in G.$$

Thus equivalent representations are "the same" up to a change of basis—they encode the same structural information about the group.

Reducibility and Irreducibility

Definition

Definition 23.3 (Invariant Subspace, Reducibility). Let $\rho : G \to \mathrm{GL}(V)$ be a representation. A subspace $W \subseteq V$ is *G-invariant* (or simply *invariant*) if

$$\rho(g)(W) \subseteq W \quad \text{for all } g \in G.$$

The representation ρ is called:

- **Irreducible** if the only invariant subspaces are $\{0\}$ and V.
- **Reducible** if there exists a proper non-trivial invariant subspace $0 \subsetneq W \subsetneq V$.
- **Completely reducible** (or *semisimple*) if V can be written as a direct sum of irreducible invariant subspaces.

Irreducible representations are the "building blocks" of representation theory. Just as we factor integers into primes or modules into simple modules, we seek to decompose representations into irreducibles.

Example

Example 23.7 (Reducible but Not Completely Reducible). Let $G = \mathbb{Z}$ and consider the representation on $\mathbb{R}^2$ given by

$$\rho(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

The subspace $W = \{(x, 0) : x \in \mathbb{R}\}$ is invariant (check: $\rho(n)(x,0)^T = (x,0)^T$). Thus ρ is reducible.

However, there is *no* complementary invariant subspace U with $\mathbb{R}^2 = W \oplus U$. Any G-invariant complement would require an eigenvector with eigenvalue 1, but $\rho(1)$ has only one-dimensional eigenspace. Thus this representation is reducible but *not* completely reducible.

This pathology cannot occur for finite groups over $\mathbb{C}$ (or any field whose characteristic doesn't divide $|G|$), as Maschke's theorem will show.

Direct Sums of Representations

Definition

Definition 23.4 (Direct Sum). Let $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ be representations. Their *direct sum* is the representation $\rho \oplus \sigma : G \to \mathrm{GL}(V \oplus W)$ defined by

$$(\rho \oplus \sigma)(g)(v, w) = (\rho(g)v, \sigma(g)w).$$

In matrix form (choosing bases for V and W):

$$(\rho \oplus \sigma)(g) = \begin{pmatrix} \rho(g) & 0 \\ 0 & \sigma(g) \end{pmatrix}.$$

A representation is completely reducible if and only if it is equivalent to a direct sum of irreducible representations.

Schur's Lemma

One of the most fundamental results in representation theory is Schur's lemma, which characterizes intertwining operators between irreducible representations.

Theorem

Theorem 23.1 (Schur's Lemma). Let $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ be irreducible representations over an algebraically closed field F (such as $\mathbb{C}$).

(a) If $\phi : V \to W$ is an intertwining operator, then either $\phi = 0$ or ϕ is an isomorphism.

(b) If $V = W$ and $\rho = \sigma$, then any intertwining operator $\phi : V \to V$ is a scalar multiple of the identity: $\phi = \lambda I$ for some $\lambda \in F$.

Proof. **(a)** Consider $\ker(\phi)$ and $\mathrm{im}(\phi)$. Since ϕ intertwines ρ and σ, both are G-invariant subspaces. Indeed, if $v \in \ker(\phi)$, then

$$\phi(\rho(g)v) = \sigma(g)\phi(v) = \sigma(g) \cdot 0 = 0,$$

so $\rho(g)v \in \ker(\phi)$. Thus $\ker(\phi) \subseteq V$ is G-invariant. Similarly, $\mathrm{im}(\phi) \subseteq W$ is G-invariant.

By irreducibility of V, either $\ker(\phi) = \{0\}$ or $\ker(\phi) = V$. If $\ker(\phi) = V$, then $\phi = 0$. If $\ker(\phi) = \{0\}$, then ϕ is injective. By irreducibility of W, either $\mathrm{im}(\phi) = \{0\}$ (impossible since ϕ injective and $V \neq \{0\}$) or $\mathrm{im}(\phi) = W$, making ϕ surjective. Thus ϕ is bijective, hence an isomorphism.

(b) Since F is algebraically closed, ϕ has an eigenvalue $\lambda \in F$. Consider $\psi = \phi - \lambda I$. Then ψ is also an intertwining operator (check: $\psi \circ \rho(g) = \rho(g) \circ \psi$), and ψ is not injective (it has λ as eigenvalue). By part (a), $\psi = 0$, so $\phi = \lambda I$. ∎

Corollary 23.2. Over $\mathbb{C}$, two irreducible representations ρ and σ are either equivalent or have no non-trivial intertwining operators.

Schur's lemma is the representation-theoretic analogue of the fact that simple modules have endomorphism rings that are division rings. Over $\mathbb{C}$, the only finite-dimensional division algebra is $\mathbb{C}$ itself, which is why we get scalars.

23.2 — Maschke's Theorem

We now arrive at one of the cornerstones of representation theory: Maschke's theorem, which guarantees complete reducibility for finite groups over fields of "good" characteristic.

Theorem

Theorem 23.3 (Maschke's Theorem). Let G be a finite group and F a field whose characteristic does not divide $|G|$. Then every representation of G over F is completely reducible.

Equivalently: If $\rho : G \to \mathrm{GL}(V)$ is a representation and $W \subseteq V$ is a G-invariant subspace, then there exists a G-invariant complement $U \subseteq V$ such that $V = W \oplus U$.

Proof. Let $W \subseteq V$ be a G-invariant subspace. We'll construct a G-invariant complement.

Step 1: Initial complement. Choose any vector space complement U_0 with $V = W \oplus U_0$ (as abstract vector spaces, this always exists). Let $\pi_0 : V \to W$ be the projection onto W along U_0. Thus:

$$\pi_0(w + u) = w \quad \text{for } w \in W, u \in U_0,$$

and $\pi_0^2 = \pi_0$, $\mathrm{im}(\pi_0) = W$, $\ker(\pi_0) = U_0$.

Step 2: Averaging over the group. The problem is that π_0 may not intertwine the G-action. We fix this by averaging:

$$\pi(v) = \frac{1}{|G|} \sum_{g \in G} \rho(g)^{-1} \pi_0 \rho(g)(v).$$

Since $\mathrm{char}(F) \nmid |G|$, division by $|G|$ is well-defined in F.

Step 3: Properties of π. We verify π is a G-equivariant projection onto W.

(π is linear): Clear from definition.

(π has image W): For $w \in W$, since W is G-invariant, $\rho(g)(w) \in W$ for all g. Thus:

$$\pi_0(\rho(g)w) = \rho(g)w$$

(projecting an element of W onto W gives itself). Therefore:

$$\pi(w) = \frac{1}{|G|} \sum_{g \in G} \rho(g)^{-1} \rho(g) w = \frac{1}{|G|} \sum_{g \in G} w = w.$$

So $\pi|_W = \mathrm{id}_W$, implying $\mathrm{im}(\pi) = W$.

(π is a projection): For any $v \in V$, $\pi(v) \in W$, so by the above, $\pi(\pi(v)) = \pi(v)$. Thus $\pi^2 = \pi$.

(π is G-equivariant): For any $h \in G$ and $v \in V$:

$$\begin{aligned}
\pi(\rho(h)v) &= \frac{1}{|G|} \sum_{g \in G} \rho(g)^{-1} \pi_0 \rho(g) \rho(h) v \\
&= \frac{1}{|G|} \sum_{g \in G} \rho(g)^{-1} \pi_0 \rho(gh) v \\
&= \frac{1}{|G|} \sum_{g' \in G} \rho(h) \rho(g')^{-1} \pi_0 \rho(g') v \quad (\text{substituting } g' = gh) \\
&= \rho(h) \pi(v).
\end{aligned}$$

The substitution works because as g ranges over G, so does $g' = gh$.

Step 4: The complement. Let $U = \ker(\pi)$. Since π is G-equivariant, U is G-invariant: if $u \in U$ and $g \in G$, then

$$\pi(\rho(g)u) = \rho(g)\pi(u) = \rho(g) \cdot 0 = 0,$$

> so $\rho(g)u \in U$. Since $\pi^2 = \pi$ and $\operatorname{im}(\pi) = W$, we have $V = W \oplus \ker(\pi) = W \oplus U$ as desired.

∎

Corollary 23.4. Every representation of a finite group over $\mathbb{C}$ is completely reducible—it is equivalent to a direct sum of irreducible representations:

$$\rho \cong \bigoplus_{i=1}^{k} \rho_i$$

where each ρ_i is irreducible.

The decomposition is not unique (the same irreducible may appear multiple times, and the order matters), but the *multiplicities* of the irreducibles are unique. This is analogous to prime factorization of integers.

Consequences for the Group Algebra

Maschke's theorem has profound implications for the structure of the group algebra $\mathbb{C}[G]$.

> **Definition**
>
> **Definition 23.5 (Group Algebra).** The *group algebra* $F[G]$ is the vector space over F with basis $\{g : g \in G\}$, equipped with multiplication extending the group operation linearly:
>
> $$\left(\sum_{g \in G} a_g g\right)\left(\sum_{h \in G} b_h h\right) = \sum_{g,h \in G} a_g b_h (gh).$$

A representation $\rho : G \to \mathrm{GL}(V)$ extends uniquely to an algebra homomorphism $\rho : F[G] \to \mathrm{End}(V)$ by linearity. Conversely, every $F[G]$-module gives a representation of G. Thus:

> **Theorem**
>
> **Theorem 23.5 (Equivalence of Categories).** The category of representations of G over F is equivalent to the category of left $F[G]$-modules.

Maschke's theorem then translates to:

Theorem

Theorem 23.6. If $\mathrm{char}(F) \nmid |G|$, the group algebra $F[G]$ is *semisimple*: every $F[G]$-module is completely reducible (a direct sum of simple modules).

By the Artin-Wedderburn theorem, a semisimple algebra is isomorphic to a direct product of matrix rings:

$$\mathbb{C}[G] \cong \prod_{i=1}^{k} \mathrm{Mat}_{n_i}(\mathbb{C})$$

where k is the number of irreducible representations of G (up to equivalence), and n_i is the dimension of the i-th irreducible representation. This gives:

Theorem

Theorem 23.7. Let G be a finite group with irreducible representations $\rho_1, \ldots, \rho_k$ over $\mathbb{C}$ of dimensions $n_1, \ldots, n_k$. Then:

$$|G| = n_1^2 + n_2^2 + \cdots + n_k^2.$$

This beautiful formula relates the group order to the dimensions of its irreducible representations!

Examples of Decomposition

Example

Example 23.8 (Decomposing the Regular Representation of S_3). Recall S_3 has three irreducible representations over $\mathbb{C}$:

- ρ_{triv}: degree 1
- ρ_{sgn}: degree 1
- ρ_{std}: degree 2

Check: $1^2 + 1^2 + 2^2 = 6 = |S_3|$. ✓

The regular representation ρ_{reg} has degree 6. By complete reducibility, it decomposes into irreducibles. The multiplicity m_i of each irreducible ρ_i in ρ_{reg} is given by the dimension formula (which we'll prove later using characters):

$$m_i = \dim(\rho_i).$$

Thus:

$$\rho_{\text{reg}} \cong \rho_{\text{triv}} \oplus \rho_{\text{sgn}} \oplus \rho_{\text{std}} \oplus \rho_{\text{std}}.$$

Check dimensions: $1 + 1 + 2 + 2 = 6$. ✓

Example

Example 23.9 (Dihedral Group D_4). The dihedral group D_4 has order 8 and five conjugacy classes. It has five irreducible representations over $\mathbb{C}$:

- Four one-dimensional representations
- One two-dimensional representation

Check: $1^2 + 1^2 + 1^2 + 1^2 + 2^2 = 8 = |D_4|$. ✓

The four one-dimensional representations come from the abelianization $D_4/[D_4, D_4] \cong C_2 \times C_2$, and the two-dimensional representation is the standard geometric action on $\mathbb{R}^2$.

When Maschke Fails

Example

Example 23.10 (Characteristic Dividing the Group Order). Consider $G = C_p = \langle g \mid g^p = e \rangle$, a cyclic group of prime order p. Let $F = \mathbb{F}_p$, the field with p elements (so $\mathrm{char}(F) = p$).

Consider the regular representation on $F[G]$, which has basis $\{1, g, g^2, \ldots, g^{p-1}\}$. Define a subspace

$$W = \mathrm{span}\{1 + g + g^2 + \cdots + g^{p-1}\}.$$

This is G-invariant: multiplying by g just cyclically permutes the sum, giving the same element. So $\dim(W) = 1$ and W is invariant.

However, there is *no* G-invariant complement to W in $F[G]$. The representation $F[G]$ is *not* completely reducible. Maschke's theorem fails because $\mathrm{char}(F) = p$ divides $|G| = p$.

When Maschke fails, the representation theory becomes much more subtle, involving *indecomposable* representations that are not irreducible but cannot be written as direct sums. This is the realm of *modular representation theory*, pioneered by Richard Brauer.

☯ Zen Meditation ☯

Complete Reducibility: The Shattered Mirror
A mirror falls and breaks into pieces.
Each piece still reflects the whole moon.
So too, a representation shatters into irreducibles,
yet each irreducible contains the essence of the group.
Breaking reveals structure; structure requires breaking.

23.3 — Character Theory

Characters provide a powerful computational tool for studying representations. They are much simpler objects than representations themselves (just functions $G \to \mathbb{C}$), yet remarkably they contain complete information about the representation up to equivalence.

Definition

Definition 23.6 (Character). Let $\rho : G \to \mathrm{GL}(V)$ be a representation. The *character* of ρ is the function $\chi : G \to \mathbb{C}$ defined by

$$\chi(g) = \mathrm{Tr}(\rho(g)),$$

the trace of the matrix $\rho(g)$ (in any basis).

Recall that for a linear transformation $T : V \to V$ with matrix representation $[T]$ in some basis, $\mathrm{Tr}(T) = \sum_i [T]_{ii}$ (sum of diagonal entries). Crucially, the trace is independent of the choice of basis: if $P[T]P^{-1}$ is the matrix in another basis, then

$$\mathrm{Tr}(P[T]P^{-1}) = \mathrm{Tr}([T]P^{-1}P) = \mathrm{Tr}([T]).$$

Therefore $\chi(g)$ is well-defined.

Basic Properties of Characters

Theorem

Theorem 23.8 (Properties of Characters). Let χ be the character of a representation $\rho : G \to \mathrm{GL}(V)$. Then:

(1) $\chi(e) = \dim(V)$ (trace of the identity is the dimension).

(2) $\chi(hgh^{-1}) = \chi(g)$ for all $g, h \in G$ (characters are *class functions*: constant on conjugacy classes).

(3) $\chi(g^{-1}) = \overline{\chi(g)}$ (complex conjugate).

(4) If ρ and σ are equivalent, then they have the same character.

(5) $\chi_{\rho \oplus \sigma} = \chi_\rho + \chi_\sigma$.

Proof. **(1)** $\rho(e) = I_V$, so $\chi(e) = \mathrm{Tr}(I_V) = \dim(V)$.

(2) Using the cyclic property of trace:

$$\chi(hgh^{-1}) = \mathrm{Tr}(\rho(h)\rho(g)\rho(h)^{-1}) = \mathrm{Tr}(\rho(g)\rho(h)^{-1}\rho(h)) = \mathrm{Tr}(\rho(g)) = \chi(g).$$

(3) For finite groups, g has finite order, so $\rho(g)$ has finite order and is diagonalizable over $\mathbb{C}$ with eigenvalues that are roots of unity: $\lambda_1, \ldots, \lambda_n$. Then $\chi(g) = \sum \lambda_i$. For g^{-1}, the eigenvalues are $\lambda_i^{-1} = \overline{\lambda_i}$ (since $|\lambda_i| = 1$). Thus $\chi(g^{-1}) = \sum \lambda_i^{-1} = \sum \overline{\lambda_i} = \overline{\sum \lambda_i} = \overline{\chi(g)}$.

(4) If $\sigma(g) = P\rho(g)P^{-1}$, then $\chi_\sigma(g) = \mathrm{Tr}(P\rho(g)P^{-1}) = \mathrm{Tr}(\rho(g)) = \chi_\rho(g)$.

(5) In block form, $(\rho \oplus \sigma)(g) = \begin{pmatrix} \rho(g) & 0 \\ 0 & \sigma(g) \end{pmatrix}$, so

$$\chi_{\rho\oplus\sigma}(g) = \mathrm{Tr}(\rho(g)) + \mathrm{Tr}(\sigma(g)) = \chi_\rho(g) + \chi_\sigma(g).$$

∎

Property (2) is crucial: characters are *class functions*, depending only on the conjugacy class of g. Since a finite group G has finitely many conjugacy classes, the character is determined by finitely many values.

Inner Product on Class Functions

Definition

Definition 23.7 (Inner Product on Class Functions). Define a Hermitian inner product on the space of class functions $f : G \to \mathbb{C}$ by

$$\langle f, g \rangle = \frac{1}{|G|} \sum_{x \in G} f(x)\overline{g(x)}.$$

This makes the space of class functions into an inner product space. The fundamental orthogonality relations show that the irreducible characters form an orthonormal basis of this space!

Theorem

Theorem 23.9 (Orthogonality Relations). Let χ_i and χ_j be the characters of irreducible representations ρ_i and ρ_j of G over $\mathbb{C}$. Then:

$$\langle \chi_i, \chi_j \rangle = \begin{cases} 1 & \text{if } \rho_i \cong \rho_j \\ 0 & \text{if } \rho_i \ncong \rho_j \end{cases} = \delta_{ij}.$$

In other words, the irreducible characters are orthonormal.

Proof. The proof uses Schur's lemma and properties of the group algebra. We sketch the key ideas.

Consider the linear map $\Phi : \mathrm{Hom}(V_i, V_j) \to \mathrm{Hom}(V_i, V_j)$ defined by

$$\Phi(T) = \frac{1}{|G|} \sum_{g \in G} \rho_j(g) T \rho_i(g)^{-1}.$$

One can show this is a projection onto the space of intertwining operators. By Schur's lemma:

$$\dim(\mathrm{Hom}_G(V_i, V_j)) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Taking traces and using properties of trace gives the orthogonality relation.

∎

A consequence of orthogonality is that representations are determined by their characters:

Corollary 23.10. Two representations ρ and σ over $\mathbb{C}$ are equivalent if and only if $\chi_\rho = \chi_\sigma$.

This is remarkable: the character, a much simpler object than the representation, contains all the essential information!

Decomposing Representations Using Characters

The orthogonality relations give us a practical tool for decomposing representations.

Theorem

Theorem 23.11 (Computing Multiplicities). Let ρ be a representation with character χ, and let $\{\rho_1, \dots, \rho_k\}$ be the irreducible representations with characters $\{\chi_1, \dots, \chi_k\}$. Then

$$\rho \cong \bigoplus_{i=1}^{k} m_i \rho_i$$

where the multiplicity m_i is given by

$$m_i = \langle \chi, \chi_i \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\chi_i(g)}.$$

Proof. From $\chi = \sum m_i \chi_i$, taking inner product with χ_j:

$$\langle \chi, \chi_j \rangle = \left\langle \sum_i m_i \chi_i, \chi_j \right\rangle = \sum_i m_i \langle \chi_i, \chi_j \rangle = \sum_i m_i \delta_{ij} = m_j.$$

∎

Theorem

Theorem 23.12 (Test for Irreducibility). A representation ρ with character χ is irreducible if and only if

$$\langle \chi, \chi \rangle = 1.$$

Proof. If $\rho \cong \bigoplus m_i \rho_i$, then $\chi = \sum m_i \chi_i$, so

$$\langle \chi, \chi \rangle = \sum_{i,j} m_i m_j \langle \chi_i, \chi_j \rangle = \sum_i m_i^2.$$

This equals 1 if and only if exactly one $m_i = 1$ and the rest are 0, i.e., ρ is irreducible. ∎

Character Tables

A *character table* is a table displaying the values of all irreducible characters on all conjugacy classes. It encodes essential information about the group's representation theory.

> **Theorem**
>
> **Theorem 23.13.** For a finite group G, the number of irreducible representations over $\mathbb{C}$ (up to equivalence) equals the number of conjugacy classes.

Thus a character table is a square matrix where rows are indexed by irreducible characters and columns by conjugacy classes.

Complete Character Table for S_3

Let's construct the complete character table for S_3. The group has three conjugacy classes:

- $C_1 = \{e\}$ with 1 element
- $C_2 = \{(12), (13), (23)\}$ with 3 elements (transpositions)
- $C_3 = \{(123), (132)\}$ with 2 elements (3-cycles)

So there are 3 irreducible representations. We know $\sum n_i^2 = 6$, and the only solution with 3 terms is $1^2 + 1^2 + 2^2 = 6$. So dimensions are 1, 1, 2.

Trivial representation χ_{triv}: $\chi(g) = 1$ for all g.

Sign representation χ_{sgn}: $\chi(\sigma) = \text{sgn}(\sigma)$.

$$\chi_{\text{sgn}}(e) = 1, \quad \chi_{\text{sgn}}((12)) = -1, \quad \chi_{\text{sgn}}((123)) = 1.$$

Standard representation χ_{std}: We compute from the matrices. For the identity, $\chi(e) = 2$ (dimension). For a transposition like (12), the matrix has trace $-1 + 1 = 0$. For a 3-cycle like (123), the trace is $0 + 1 = -1$.

The complete character table:

S_3	C_1 **(1 elem)**	C_2 **(3 elem)**	C_3 **(2 elem)**
χ_{triv}	1	1	1
χ_{sgn}	1	-1	1
χ_{std}	2	0	-1

We can verify orthogonality. For instance:

$$\langle \chi_{\text{triv}}, \chi_{\text{sgn}} \rangle = \frac{1}{6}(1 \cdot 1 \cdot 1 + 3 \cdot 1 \cdot (-1) + 2 \cdot 1 \cdot 1) = \frac{1}{6}(1 - 3 + 2) = 0. \quad \checkmark$$

Character Table for D_4

The dihedral group $D_4 = \langle r, s \mid r^4 = s^2 = e, srs = r^{-1} \rangle$ has order 8 and five conjugacy classes:

- $C_1 = \{e\}$
- $C_2 = \{r^2\}$
- $C_3 = \{r, r^3\}$
- $C_4 = \{s, r^2 s\}$
- $C_5 = \{rs, r^3 s\}$

There are 5 irreducible representations. Since $\sum n_i^2 = 8$ and we have 5 terms, the only solution is $1^2 + 1^2 + 1^2 + 1^2 + 2^2 = 8$. So four are one-dimensional and one is two-dimensional.

The four one-dimensional characters come from homomorphisms $D_4 \to \mathbb{C}^*$, which factor through the abelianization $D_4/[D_4, D_4] \cong C_2 \times C_2$. This gives four characters determined by the values on r and s.

D_4	e	r^2	$\{r, r^3\}$	$\{s, r^2 s\}$	$\{rs, r^3 s\}$
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

The two-dimensional representation χ_5 is the standard geometric action on $\mathbb{R}^2$.

Character Table for A_4

The alternating group A_4 has order 12 and four conjugacy classes:

- $C_1 = \{e\}$
- $C_2 = \{(12)(34), (13)(24), (14)(23)\}$ — double transpositions (3 elements)
- $C_3 = \{(123), (142), (134), (243)\}$ — 3-cycles of type (123) (4 elements)
- $C_4 = \{(132), (124), (143), (234)\}$ — 3-cycles of type (132) (4 elements)

From $\sum n_i^2 = 12$ with 4 irreducibles, we get $1^2 + 1^2 + 1^2 + 3^2 = 12$.

A_4	e	**double trans**	**type (123)**	**type (132)**
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	3	-1	0	0

Here $\omega = e^{2\pi i/3}$ is a primitive cube root of unity. The three one-dimensional characters come from $A_4/V_4 \cong C_3$ where $V_4 = \{e, (12)(34), (13)(24), (14)(23)\}$ is the Klein four-group.

Example

Example 23.11 (Determining Irreducibility). Consider a representation ρ of S_3 with character values:

$$\chi(e) = 3, \quad \chi((12)) = 1, \quad \chi((123)) = 0.$$

Is ρ irreducible? Compute:

$$\langle \chi, \chi \rangle = \frac{1}{6}(1 \cdot 3^2 + 3 \cdot 1^2 + 2 \cdot 0^2) = \frac{1}{6}(9 + 3) = 2.$$

Since $\langle \chi, \chi \rangle = 2 \neq 1$, ρ is *not* irreducible.

Example

Example 23.12 (Decomposing a Representation). Using the character from Example 23.11, decompose ρ into irreducibles of S_3.

Compute multiplicities:

$$\begin{aligned} m_{\text{triv}} &= \langle \chi, \chi_{\text{triv}} \rangle = \frac{1}{6}(1 \cdot 3 \cdot 1 + 3 \cdot 1 \cdot 1 + 2 \cdot 0 \cdot 1) = \frac{6}{6} = 1 \\ m_{\text{sgn}} &= \langle \chi, \chi_{\text{sgn}} \rangle = \frac{1}{6}(1 \cdot 3 \cdot 1 + 3 \cdot 1 \cdot (-1) + 2 \cdot 0 \cdot 1) = \frac{0}{6} = 0 \\ m_{\text{std}} &= \langle \chi, \chi_{\text{std}} \rangle = \frac{1}{6}(1 \cdot 3 \cdot 2 + 3 \cdot 1 \cdot 0 + 2 \cdot 0 \cdot (-1)) = \frac{6}{6} = 1 \end{aligned}$$

Therefore:

$$\rho \cong \chi_{\text{triv}} \oplus \chi_{\text{std}}.$$

Check: $\dim(\rho) = 1 + 2 = 3$. ✓

Example

Example 23.13 (Tensor Products). The *tensor product* of representations $\rho \otimes \sigma$ has character $\chi_{\rho\otimes\sigma}(g) = \chi_\rho(g)\chi_\sigma(g)$.

For S_3, compute $\chi_{\text{std}} \otimes \chi_{\text{std}}$:

$$\begin{aligned}(\chi_{\text{std}} \otimes \chi_{\text{std}})(e) &= 2 \cdot 2 = 4\\ (\chi_{\text{std}} \otimes \chi_{\text{std}})((12)) &= 0 \cdot 0 = 0\\ (\chi_{\text{std}} \otimes \chi_{\text{std}})((123)) &= (-1)(-1) = 1\end{aligned}$$

Decompose this 4-dimensional representation:

$$\begin{aligned}m_{\text{triv}} &= \frac{1}{6}(1\cdot 4\cdot 1 + 3\cdot 0\cdot 1 + 2\cdot 1\cdot 1) = 1\\ m_{\text{sgn}} &= \frac{1}{6}(1\cdot 4\cdot 1 + 3\cdot 0\cdot (-1) + 2\cdot 1\cdot 1) = 1\\ m_{\text{std}} &= \frac{1}{6}(1\cdot 4\cdot 2 + 3\cdot 0\cdot 0 + 2\cdot 1\cdot (-1)) = 1\end{aligned}$$

Thus:

$$\chi_{\text{std}} \otimes \chi_{\text{std}} \cong \chi_{\text{triv}} \oplus \chi_{\text{sgn}} \oplus \chi_{\text{std}}.$$

⚡ **Greek Meditation** ⚡

The Character: Essence Beyond Appearance

Plato taught that Forms are known not by sensation but by reason.
So too, the character captures the essence of representation—
Not the matrices themselves (mere shadows on the cave wall),
but the trace, the invariant, the eternal aspect.
To know the character is to know the Form.

23.4 — Applications to Physics and Chemistry

Burnside's Theorem

Theorem

Theorem 23.14 (Burnside's p^aq^b Theorem). Let G be a finite group whose order is $|G| = p^aq^b$ where p and q are primes. Then G is solvable.

This remarkable theorem, proved by Burnside using character theory, was a major milestone. It shows that groups of certain orders cannot be "simple" (except in trivial cases). The proof uses deep properties of characters and shows the power of representation theory in pure group theory.

Applications to Chemistry: Molecular Symmetry

Representation theory provides the mathematical framework for understanding molecular vibrations and spectroscopy.

Example

Example 23.14 (Water Molecule H_2O — C_{2v} Symmetry). The water molecule has symmetry group C_{2v} with four elements:

- E: identity
- C_2: rotation by 180° about axis through O
- σ_v: reflection through molecular plane
- σ_v': reflection through perpendicular plane

The character table for C_{2v}:

C_{2v}	E	C_2	σ_v	σ_v'
A_1	1	1	1	1
A_2	1	1	−1	−1
B_1	1	−1	1	−1
B_2	1	−1	−1	1

The three vibrational modes of H_2O decompose as:

$$\Gamma_{\text{vib}} = 2A_1 \oplus B_2.$$

This predicts three normal modes: two symmetric stretches (A_1) and one asymmetric stretch (B_2). Representation theory determines which modes are infrared-active and Raman-active!

Example

Example 23.15 (Benzene — D_{6h} Symmetry). Benzene (C_6H_6) has the highly symmetric dihedral group D_{6h} with 24 elements. The character table has 12 irreducible representations. Using representation theory, chemists can:

- Classify the 30 vibrational modes
- Predict IR and Raman spectra
- Understand molecular orbitals
- Explain aromaticity and bonding

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/home/ubuntu/abstract_algebra_textbook/volume_5/chapter_23_representation_theory.html

Applications to Physics: Quantum Mechanics

Symmetry lies at the heart of quantum mechanics, and representation theory is the mathematical language of symmetry.

Example

Example 23.16 (Angular Momentum and SO(3)). The rotation group SO(3) in three dimensions plays a fundamental role in quantum mechanics. The irreducible representations of SO(3) are classified by a non-negative integer or half-integer ℓ (the *angular momentum quantum number*), with dimension $2\ell + 1$.

These representations describe:

- $\ell = 0$: *s*-orbitals (spherically symmetric)
- $\ell = 1$: *p*-orbitals (three-dimensional)
- $\ell = 2$: *d*-orbitals (five-dimensional)
- $\ell = 1/2$: Spin-½ particles (electrons, quarks)

The "ladder operators" $L_\pm$ in quantum mechanics are manifestations of the representation theory of the Lie algebra $\mathfrak{so}(3)$.

Example

Example 23.17 (Particle Physics and the Standard Model). The Standard Model of particle physics is fundamentally a theory of representations of the gauge group

$$SU(3) \times SU(2) \times U(1).$$

Elementary particles are classified by how they transform—i.e., which representations they belong to:

- Quarks: fundamental representation of SU(3) (color charge)
- Leptons: singlets under SU(3)
- Gluons: adjoint representation of SU(3) (8-dimensional)
- W and Z bosons: from SU(2)

Representation theory determines allowed interactions, conserved quantities, and the structure of the universe at the most fundamental level!

Selection Rules and Wigner-Eckart Theorem

Representation theory explains why certain quantum transitions are "forbidden."

Theorem

Theorem 23.15 (Selection Rules). A matrix element $\langle \psi_f | \hat{O} | \psi_i \rangle$ vanishes unless the tensor product $\rho_f \otimes \rho_O$ contains ρ_i in its decomposition, where ρ_i, ρ_f, ρ_O are the representations transforming ψ_i, ψ_f, and $\hat{O}$.

This is why, for example, electric dipole transitions obey $\Delta \ell = \pm 1$ (angular momentum can only change by one unit).

23.5 — Representations of Lie Algebras

While we've focused on finite groups, representation theory extends beautifully to Lie groups and their associated Lie algebras. We give a brief introduction focusing on $\mathfrak{sl}_2$, which contains the essence of the general theory.

Definition

Definition 23.8 (Lie Algebra). A *Lie algebra* over a field F is a vector space $\mathfrak{g}$ equipped with a bilinear operation $[-,-]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ (the *Lie bracket*) satisfying:

(1) Antisymmetry: $[X,Y]=-[Y,X]$ for all $X,Y\in\mathfrak{g}$.

(2) Jacobi identity: $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$ for all $X,Y,Z\in\mathfrak{g}$.

Example

Example 23.18 (Classical Lie Algebras).

(1) $\mathfrak{gl}_n$: The space of all $n\times n$ matrices with bracket $[X,Y]=XY-YX$ (the commutator).

(2) $\mathfrak{sl}_n$: The space of $n\times n$ matrices with trace zero, with the commutator bracket. This is the Lie algebra of the special linear group SL_n.

(3) $\mathfrak{so}_n$: The space of $n\times n$ skew-symmetric matrices ($A^T=-A$), the Lie algebra of the special orthogonal group SO_n.

Definition

Definition 23.9 (Representation of a Lie Algebra). A *representation* of a Lie algebra $\mathfrak{g}$ on a vector space V is a Lie algebra homomorphism

$$\rho:\mathfrak{g}\to\mathfrak{gl}(V),$$

i.e., a linear map preserving the bracket: $\rho([X,Y])=[\rho(X),\rho(Y)]$.

The Lie Algebra $\mathfrak{sl}_2$

The Lie algebra $\mathfrak{sl}_2$ is three-dimensional with standard basis:

$$e=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad f=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad h=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

satisfying the commutation relations:

$$[h,e]=2e,\quad [h,f]=-2f,\quad [e,f]=h.$$

These relations encode the essence of $\mathfrak{sl}_2$ and reappear throughout mathematics and physics (e.g., as raising and lowering operators in quantum mechanics).

Theorem

Theorem 23.16 (Classification of Irreducible $\mathfrak{sl}_2$-Representations). Every finite-dimensional irreducible representation of $\mathfrak{sl}_2$ over $\mathbb{C}$ is, up to isomorphism, of the form V_n for some $n \geq 0$, where:

- $\dim(V_n) = n + 1$
- V_n has a basis $\{v_0, v_1, \ldots, v_n\}$ (weight vectors)
- $h \cdot v_k = (n - 2k)v_k$ (eigenvalues of h)
- $e \cdot v_k = (k + 1)v_{k+1}$ (raising operator, with $e \cdot v_n = 0$)
- $f \cdot v_k = (n - k + 1)v_{k-1}$ (lowering operator, with $f \cdot v_0 = 0$)

Proof. (Sketch) Let V be an irreducible $\mathfrak{sl}_2$-representation. The element h is diagonalizable (over $\mathbb{C}$), so V decomposes into *weight spaces*:

$$V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda, \quad V_\lambda = \{v \in V : h \cdot v = \lambda v\}.$$

From $[h, e] = 2e$, one shows that e raises the weight by 2: if $v \in V_\lambda$, then $e \cdot v \in V_{\lambda+2}$. Similarly, f lowers the weight by 2: $f \cdot v \in V_{\lambda-2}$.

Since V is finite-dimensional, this "ladder" of weights must terminate. There exists a *highest weight vector* v_0 with $e \cdot v_0 = 0$ and $h \cdot v_0 = nv_0$ for some $n \in \mathbb{Z}_{\geq 0}$. The vectors

$$v_0, f \cdot v_0, f^2 \cdot v_0, \ldots, f^n \cdot v_0$$

span an $(n+1)$-dimensional irreducible representation V_n, and by irreducibility, $V = V_n$. One can compute the action explicitly and show all irreducibles have this form.

∎

Example

Example 23.19 (The Representation V_1). This is 2-dimensional with basis $\{v_0, v_1\}$. The action is:

$$\begin{aligned} h \cdot v_0 &= v_0, & h \cdot v_1 &= -v_1, \\ e \cdot v_0 &= 0, & e \cdot v_1 &= v_0, \\ f \cdot v_0 &= v_1, & f \cdot v_1 &= 0. \end{aligned}$$

In matrix form (with respect to basis $\{v_0, v_1\}$):

$$\rho(e) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \rho(f) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \rho(h) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This is just the defining representation!

Connection to Physics: Spin and Angular Momentum

The representations V_n describe particles with spin $n/2$:

- V_0: spin-0 particles (scalar bosons, e.g., Higgs)
- V_1: spin-½ particles (fermions: electrons, quarks)
- V_2: spin-1 particles (vector bosons: photon, W, Z, gluons)
- V_3: spin-3/2 particles (hypothetical gravitino)
- V_4: spin-2 particles (graviton)

The operators e and f are the *ladder operators* (raising and lowering operators) familiar from quantum mechanics. The eigenvalues of h are the m quantum numbers ($m = \ell, \ell - 1, \ldots, -\ell$ for angular momentum ℓ).

Highest Weight Theory

The classification of $\mathfrak{sl}_2$-representations generalizes beautifully to all semisimple Lie algebras:

Theorem

Theorem 23.17 (Highest Weight Theorem). Finite-dimensional irreducible representations of a semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ are classified by their *highest weights*—elements of a certain lattice in the dual of a Cartan subalgebra.

This deep theorem, due to Élie Cartan and Hermann Weyl, is one of the crowning achievements of 20th century mathematics. It unifies representation theory with geometry

(via root systems and Dynkin diagrams) and has profound applications in physics, algebraic geometry, and number theory.

Practice Problems

Problem 23.1. Show that the regular representation of any finite group G is faithful (i.e., $\rho_{\text{reg}}(g) = \rho_{\text{reg}}(h)$ implies $g = h$).

> ***Solution.*** Suppose $\rho_{\text{reg}}(g) = \rho_{\text{reg}}(h)$. For any $x \in G$, we have
>
> $$\rho_{\text{reg}}(g)(e_x) = e_{gx} = e_{hx} = \rho_{\text{reg}}(h)(e_x).$$
>
> In particular, taking $x = e$, we get $e_g = e_h$, which implies $g = h$. Thus ρ_{reg} is injective, hence faithful.

Problem 23.2. Find all two-dimensional representations of the cyclic group $C_4 = \langle g \mid g^4 = e \rangle$ over $\mathbb{C}$.

> ***Solution.*** A representation $\rho : C_4 \to \mathrm{GL}_2(\mathbb{C})$ is determined by $\rho(g)$, with $\rho(g)^4 = I_2$. So $\rho(g)$ is a 2×2 matrix satisfying $A^4 = I$. The matrix must be diagonalizable with eigenvalues that are 4th roots of unity: $1, -1, i, -i$.
>
> The possible representations are:
>
> 1. Direct sums of one-dimensional representations: $\rho_j \oplus \rho_k$ where $\rho_j(g) = \omega^j$ and $\omega = e^{2\pi i/4} = i$. These give matrices like $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$, etc.
> 2. Non-diagonalizable representations with Jordan blocks (but over $\mathbb{C}$ and for finite order, these reduce to diagonal).
>
> Over $\mathbb{C}$, every two-dimensional representation of C_4 is a direct sum of two one-dimensional representations. There are no irreducible 2-dimensional representations.

Problem 23.3. Compute the character table for the Klein four-group $V_4 = \{e, a, b, ab\}$ where $a^2 = b^2 = (ab)^2 = e$.

> ***Solution.*** The group $V_4 \cong C_2 \times C_2$ is abelian with 4 conjugacy classes (each element is its own class). Thus there are 4 irreducible representations, and since $\sum n_i^2 = 4$, all are one-dimensional.
>
> The character table:
>
> | V_4 | e | a | b | ab |
> |---|---|---|---|---|
> | χ_1 | 1 | 1 | 1 | 1 |
> | χ_2 | 1 | 1 | -1 | -1 |

χ_3 1 −1 1 −1

χ_4 1 −1 −1 1

These come from group homomorphisms $V_4 \to \{\pm 1\}$, which are determined by the images of a and b.

Problem 23.4. Let V be an irreducible representation of G over $\mathbb{C}$. Show that the only G-invariant subspaces of V are $\{0\}$ and V.

Solution. This is by definition of irreducibility!

Problem 23.5. Prove that equivalent representations have the same character.

Solution. If ρ and σ are equivalent, there exists an invertible P such that $\sigma(g) = P\rho(g)P^{-1}$ for all g. Then:

$$\chi_\sigma(g) = \operatorname{Tr}(\sigma(g)) = \operatorname{Tr}(P\rho(g)P^{-1}) = \operatorname{Tr}(\rho(g)P^{-1}P) = \operatorname{Tr}(\rho(g)) = \chi_\rho(g).$$

Problem 23.6. Show that for any representation ρ of a finite group G, $\chi_\rho(g^{-1}) = \overline{\chi_\rho(g)}$.

Solution. Since $|G| < \infty$, each element g has finite order, so $\rho(g)$ has finite order and is diagonalizable over $\mathbb{C}$ with eigenvalues that are roots of unity: $\lambda_1, \dots, \lambda_n$ with $|\lambda_i| = 1$.

Then $\chi_\rho(g) = \sum_i \lambda_i$. For g^{-1}, the eigenvalues are λ_i^{-1}. Since $|\lambda_i| = 1$, we have $\lambda_i^{-1} = \overline{\lambda_i}$. Thus:

$$\chi_\rho(g^{-1}) = \sum_i \lambda_i^{-1} = \sum_i \overline{\lambda_i} = \overline{\sum_i \lambda_i} = \overline{\chi_\rho(g)}.$$

Problem 23.7. Using the character table of S_3 to decompose the regular representation into irreducibles.

Solution. The character of ρ_{reg} is:

$$\chi_{\text{reg}}(g) = \begin{cases} 6 & \text{if } g = e \\ 0 & \text{if } g \neq e \end{cases}$$

(only identity fixes any basis element).

Compute multiplicities:

$$m_{\text{triv}} = \frac{1}{6}(1 \cdot 6 \cdot 1) = 1$$
$$m_{\text{sgn}} = \frac{1}{6}(1 \cdot 6 \cdot 1) = 1$$
$$m_{\text{std}} = \frac{1}{6}(1 \cdot 6 \cdot 2) = 2$$

Thus $\rho_{\text{reg}} \cong \chi_{\text{triv}} \oplus \chi_{\text{sgn}} \oplus 2\chi_{\text{std}}$.

Problem 23.8. For the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, determine the number and dimensions of irreducible representations over $\mathbb{C}$.

> ***Solution.*** The group Q_8 has order 8 and 5 conjugacy classes:
>
> $$\{1\}, \{-1\}, \{\pm i\}, \{\pm j\}, \{\pm k\}.$$
>
> So there are 5 irreducible representations.
>
> Since $\sum n_i^2 = 8$ with 5 terms, the only solution is $1^2 + 1^2 + 1^2 + 1^2 + 2^2 = 8$. Thus four irreducibles have dimension 1 and one has dimension 2.

Problem 23.9. Show that the number of irreducible representations of a finite abelian group G equals $|G|$.

> ***Solution.*** An abelian group has $|G|$ conjugacy classes (each element is its own class). The number of irreducible representations equals the number of conjugacy classes. Thus there are $|G|$ irreducibles.
>
> Since $\sum n_i^2 = |G|$ and there are $|G|$ irreducibles, we must have $n_i = 1$ for all i. Thus every irreducible representation is one-dimensional, i.e., a homomorphism $G \to \mathbb{C}^*$.

Problem 23.10. Prove Schur's Lemma part (a): If $\phi : V \to W$ intertwines two irreducible representations ρ and σ, then either $\phi = 0$ or ϕ is an isomorphism.

> ***Solution.*** See proof of Theorem 23.1 on page 6.

Problem 23.11. Find all homomorphisms $D_3 = S_3 \to \mathbb{C}^*$ (i.e., all one-dimensional representations).

> ***Solution.*** A homomorphism $\phi : S_3 \to \mathbb{C}^*$ must send generators (12) and (123) to elements satisfying the relations:
>
> $$\phi((12))^2 = 1, \quad \phi((123))^3 = 1, \quad \phi((12)(123)(12)) = \phi((123))^{-1}.$$
>
> So $\phi((12)) \in \{\pm 1\}$ and $\phi((123)) \in \{1, \omega, \omega^2\}$ where $\omega = e^{2\pi i/3}$. But the last relation shows $\phi((123)) = \phi((123))^{-1}$, so $\phi((123))^2 = 1$, implying $\phi((123)) = \pm 1$.
>
> Since $(12)(123)(12) = (132) = (123)^{-1}$ in S_3, we need $\phi((123)) = \phi((123))^{-1}$, which gives $\phi((123)) = \pm 1$. Combined with $(123)^3 = e$, we get $\phi((123)) = 1$.
>
> Thus there are two homomorphisms: the trivial one and the sign homomorphism.

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Problem 23.12. Show that $\chi_{\rho \oplus \sigma} = \chi_\rho + \chi_\sigma$ for any representations ρ and σ.

Solution. In block form, $(\rho \oplus \sigma)(g) = \begin{pmatrix} \rho(g) & 0 \\ 0 & \sigma(g) \end{pmatrix}$. Taking trace:

$$\chi_{\rho\oplus\sigma}(g) = \operatorname{Tr}\begin{pmatrix} \rho(g) & 0 \\ 0 & \sigma(g) \end{pmatrix} = \operatorname{Tr}(\rho(g)) + \operatorname{Tr}(\sigma(g)) = \chi_\rho(g) + \chi_\sigma(g).$$

Problem 23.13. Verify the orthogonality relation $\langle \chi_{\text{triv}}, \chi_{\text{std}} \rangle = 0$ for S_3.

Solution. Using the character table:

$$\langle \chi_{\text{triv}}, \chi_{\text{std}} \rangle = \frac{1}{6}(1 \cdot 1 \cdot 2 + 3 \cdot 1 \cdot 0 + 2 \cdot 1 \cdot (-1)) = \frac{1}{6}(2 + 0 - 2) = 0.$$

Problem 23.14. Let G be a group with 3 irreducible representations of dimensions 1, 2, and n. Find n and $|G|$.

Solution. We have $1^2 + 2^2 + n^2 = |G|$, so $5 + n^2 = |G|$. Since $|G|$ must be a positive integer, $n^2 = |G| - 5$. The smallest possibilities:

- $n = 1$: $|G| = 6$ (this is S_3)
- $n = 2$: $|G| = 9$
- $n = 3$: $|G| = 14$

The most common case is $n = 1, |G| = 6$.

Problem 23.15. Show that any representation of a finite group over $\mathbb{C}$ is equivalent to a unitary representation (one where all $\rho(g)$ are unitary matrices).

Solution. Start with any representation ρ on V with an inner product $\langle -, - \rangle_0$. Define a new inner product by averaging:

$$\langle v, w \rangle = \frac{1}{|G|} \sum_{g \in G} \langle \rho(g)v, \rho(g)w \rangle_0.$$

This new inner product is G-invariant:

$$\langle \rho(h)v, \rho(h)w \rangle = \frac{1}{|G|} \sum_{g \in G} \langle \rho(gh)v, \rho(gh)w \rangle_0 = \langle v, w \rangle.$$

(The sum over gh as g varies equals the sum over all group elements.)

With respect to this inner product, all $\rho(g)$ are unitary.

Problem 23.16. Explain why Maschke's theorem fails for $G = C_2$ over $\mathbb{F}_2$.

Solution. Since $\operatorname{char}(\mathbb{F}_2) = 2$ divides $|C_2| = 2$, division by $|G|$ is not well-defined in $\mathbb{F}_2$ (we have $2 = 0$ in $\mathbb{F}_2$). The averaging trick in the proof of Maschke's theorem requires dividing by $|G|$, which fails.

Concretely, consider $\mathbb{F}_2[C_2]$ with basis $\{1, g\}$. The subspace $W = \text{span}\{1+g\}$ is C_2-invariant (multiplying by g gives $g(1+g) = g + g^2 = g + 1 = 1 + g$). But there is no C_2-invariant complement to W.

Problem 23.17. Compute the character of the tensor product $\chi_{\text{std}} \otimes \chi_{\text{sgn}}$ for S_3 and decompose it.

Solution. For tensor product, $\chi_{\rho\otimes\sigma}(g) = \chi_\rho(g)\chi_\sigma(g)$. So:

$$\begin{aligned}(\chi_{\text{std}} \otimes \chi_{\text{sgn}})(e) &= 2 \cdot 1 = 2\\ (\chi_{\text{std}} \otimes \chi_{\text{sgn}})((12)) &= 0 \cdot (-1) = 0\\ (\chi_{\text{std}} \otimes \chi_{\text{sgn}})((123)) &= (-1) \cdot 1 = -1\end{aligned}$$

This is exactly χ_{std}! Check orthogonality:

$$\langle \chi_{\text{std}} \otimes \chi_{\text{sgn}}, \chi_{\text{std}} \rangle = \frac{1}{6}(1 \cdot 4 + 0 + 2 \cdot 1) = 1.$$

So $\chi_{\text{std}} \otimes \chi_{\text{sgn}} \cong \chi_{\text{std}}$. Tensoring with the sign representation leaves χ_{std} unchanged!

Problem 23.18. For any cyclic group C_n, show that all irreducible representations over $\mathbb{C}$ are one-dimensional.

Solution. Since C_n is abelian, it has n conjugacy classes (one for each element). Thus there are n irreducible representations.

Since $\sum d_i^2 = n$ with n terms, we must have $d_i = 1$ for all i. Every irreducible is one-dimensional.

Explicitly, they are $\rho_k(g^j) = \omega_k^j$ where $\omega_k = e^{2\pi i k/n}$ for $k = 0, 1, \ldots, n-1$.

Problem 23.19. Show that for $\mathfrak{sl}_2$, the commutator $[e, f] = ef - fe$ equals h.

Solution. Compute directly:

$$ef = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$$

$$fe = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$$

$$[e, f] = ef - fe = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} - \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = h.$$

Problem 23.20. (Challenge) Prove that the character of the regular representation satisfies:

$$\chi_{\text{reg}}(g) = \begin{cases}|G| & \text{if } g = e\\ 0 & \text{if } g \neq e\end{cases}$$

> ***Solution.*** The regular representation has basis $\{e_h : h \in G\}$ with $\rho_{\text{reg}}(g)(e_h) = e_{gh}$. In matrix form, the entry $[\rho_{\text{reg}}(g)]_{h_1,h_2}$ is 1 if $gh_2 = h_1$ and 0 otherwise.
>
> The trace counts diagonal entries: $[\rho_{\text{reg}}(g)]_{h,h} = 1$ iff $gh = h$ iff $g = e$.
>
> - If $g = e$: All diagonal entries are 1, so $\chi_{\text{reg}}(e) = |G|$.
> - If $g \neq e$: No diagonal entries are 1 (since $gh = h$ implies $g = e$), so $\chi_{\text{reg}}(g) = 0$.

> **☯ Zen Meditation ☯**
>
> **The Character: Essence Beyond Appearance**
>
> A stone Buddha sits in the garden.
> Rain falls, snow falls, sun shines.
> The form changes, yet the Buddha-nature remains.
>
> So too, a representation may take infinite forms—
> different bases, different matrices, different realizations.
> Yet the character, the trace, captures the essence.
>
> To know the character is to know the representation.
> To know the representation is to know the group.
> To know the group is to know the symmetry.
> To know the symmetry is to know the world.
>
> Form is emptiness; emptiness is form.
> Representation is character; character is representation.
>
> The thousand and one reflections in the dewdrops—
> all reveal the same moon.

"In mathematics, you don't understand things.
You just get used to them."
— John von Neumann

⚘ End of Chapter 23 ⚘

CHAPTER TWENTY-FOUR

Chapter 24: Algebraic Number Theory

"God created the integers, all else is the work of man."
— Leopold Kronecker

In which we extend the integers beyond $\mathbb{Z}$,
discover that unique factorization sometimes fails,
and witness how ideals restore order to chaos—
culminating in the proof of Fermat's Last Theorem.

Chapter Overview:

- **24.1** — Algebraic Integers
- **24.2** — Unique Factorization and Ideal Theory
- **24.3** — Ramification Theory
- **24.4** — Class Field Theory
- **24.5** — Fermat's Last Theorem

Chapter 24 — Algebraic Number Theory

Introduction: Beyond the Integers

Algebraic number theory stands as one of the most profound and beautiful areas of mathematics, combining the elementary arithmetic of the integers with the sophisticated machinery of modern algebra. Born from attempts to prove Fermat's Last Theorem and understand the nature of prime numbers, it has evolved into a vast theoretical edifice with connections to geometry, analysis, and mathematical physics.

The fundamental idea is deceptively simple: *extend the integers*. Just as we extended $\mathbb{Q}$ to obtain algebraic number fields $K = \mathbb{Q}(\alpha)$, we can ask: what is the "right" generalization of $\mathbb{Z}$ inside K? The answer—the ring of algebraic integers $\mathcal{O}_K$—shares many properties with $\mathbb{Z}$ but reveals surprising new phenomena.

Chief among these surprises is the failure of unique factorization. In $\mathbb{Z}$, every integer factors uniquely into primes (up to order and units). But in $\mathbb{Z}[\sqrt{-5}]$, we

have

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$$

—two genuinely different factorizations into irreducibles! This crisis motivated Kummer and Dedekind to develop the theory of ideals, which restores unique factorization at a higher level: not elements, but *ideals* factor uniquely.

> ⚡ **Greek Meditation** ⚡
>
> **Number Fields: The Pythagorean Dream**
> The Pythagoreans believed "all is number"—
> that number is the essence of reality itself.
> But what is a number?
> $\mathbb{Z}$ extends to $\mathbb{Q}$ extends to $\mathbb{R}$ extends to $\mathbb{C}$—
> each extension revealing new structure, new harmony.
> Algebraic number theory explores the middle ground:
> extensions of $\mathbb{Q}$ that preserve algebraic structure,
> numbers born not from limits but from polynomial equations.
> Here the Pythagorean dream takes algebraic form.

The theory we develop has profound applications. Ramification theory explains how primes "split" in extensions—a phenomenon that governs everything from the solvability of equations to the distribution of primes. Class field theory provides a breathtaking connection between Galois groups and arithmetic, generalizing quadratic reciprocity. And the tools of algebraic number theory were essential in Andrew Wiles' proof of Fermat's Last Theorem, one of the greatest intellectual achievements in human history.

We begin with the definition and basic properties of algebraic integers, establish the failure of unique factorization and its resolution via ideals, explore how primes decompose in extensions, and conclude with the story of Fermat's Last Theorem.

24.1 — Algebraic Integers

Definition

Definition 24.1 (Algebraic Integer). A complex number $\alpha \in \mathbb{C}$ is an *algebraic integer* if it is a root of a monic polynomial with integer coefficients, i.e., if there exists

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$$

with $f(\alpha) = 0$.

This generalizes the usual integers: every $n \in \mathbb{Z}$ is an algebraic integer (it satisfies $x - n = 0$). But many other numbers qualify as well.

Example

Example 24.1 (Basic Examples).

(1) All $n \in \mathbb{Z}$: satisfies $x - n = 0$.

(2) $\sqrt{2}$: satisfies $x^2 - 2 = 0$.

(3) $\sqrt[3]{3}$: satisfies $x^3 - 3 = 0$.

(4) $i = \sqrt{-1}$: satisfies $x^2 + 1 = 0$.

(5) $\omega = e^{2\pi i/3}$: a primitive cube root of unity, satisfies $x^3 - 1 = (x-1)(x^2 + x + 1) = 0$, so $\omega^2 + \omega + 1 = 0$.

(6) The golden ratio $\phi = \frac{1+\sqrt{5}}{2}$: satisfies $x^2 - x - 1 = 0$ (check: $\phi^2 = \phi + 1$).

Example

Example 24.2 (Non-Examples).

(1) $1/2$: Suppose $1/2$ satisfies a monic polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x]$. Then:

$$\left(\frac{1}{2}\right)^n + a_{n-1}\left(\frac{1}{2}\right)^{n-1} + \cdots + a_0 = 0.$$

Multiplying by 2^n:

$$1 + 2a_{n-1} + 4a_{n-2} + \cdots + 2^n a_0 = 0.$$

The left side is odd, contradicting that it equals 0. So $1/2$ is not an algebraic integer.

(2) $\sqrt{1/2}$: satisfies $x^2 - 1/2 = 0$, but this is not monic with integer coefficients. If $\sqrt{1/2}$ were an algebraic integer, the same argument as above would give a contradiction.

The Ring of Integers in a Number Field

Definition

Definition 24.2 (Number Field, Ring of Integers). A *number field* is a finite extension $K/\mathbb{Q}$. The *ring of integers* of K, denoted $\mathcal{O}_K$, is

$$\mathcal{O}_K = \{\alpha \in K : \alpha \text{ is an algebraic integer}\}.$$

Theorem

Theorem 24.1. For any number field K, $\mathcal{O}_K$ is a ring.

Proof. (Sketch) If $\alpha, \beta \in \mathcal{O}_K$, we must show $\alpha + \beta, \alpha\beta \in \mathcal{O}_K$. This is non-trivial! The key is to show that α is an algebraic integer if and only if $\mathbb{Z}[\alpha]$ is a finitely generated abelian group. Then $\mathbb{Z}[\alpha, \beta]$ is finitely generated, implying $\alpha + \beta$ and $\alpha\beta$ are algebraic integers. ∎

Example

Example 24.3 (Rings of Integers — Explicit Computations).

(1) $\mathcal{O}_{\mathbb{Q}} = \mathbb{Z}$: The only rational numbers that are algebraic integers are the integers themselves.

(2) $\mathcal{O}_{\mathbb{Q}(i)} = \mathbb{Z}[i]$: The *Gaussian integers*. An element $a + bi \in \mathbb{Q}(i)$ is an algebraic integer iff $a, b \in \mathbb{Z}$. To see this: $a + bi$ satisfies

$$(x - (a + bi))(x - (a - bi)) = x^2 - 2ax + (a^2 + b^2) = 0.$$

This is monic with integer coefficients iff $2a \in \mathbb{Z}$ and $a^2 + b^2 \in \mathbb{Z}$. For rational a, b, this forces $a, b \in \mathbb{Z}$.

(3) $\mathcal{O}_{\mathbb{Q}(\sqrt{-5})} = \mathbb{Z}[\sqrt{-5}]$: Similar analysis. An element $a + b\sqrt{-5}$ with $a, b \in \mathbb{Q}$ is an algebraic integer iff $a, b \in \mathbb{Z}$.

(4) $\mathcal{O}_{\mathbb{Q}(\sqrt{5})} = \mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$: Here something subtle happens! While $\mathbb{Z}[\sqrt{5}] \subseteq \mathcal{O}_{\mathbb{Q}(\sqrt{5})}$, the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$ is also an algebraic integer (it satisfies $x^2 - x - 1 = 0$). In fact:

$$\mathcal{O}_{\mathbb{Q}(\sqrt{5})} = \mathbb{Z}[\phi] = \{a + b\phi : a, b \in \mathbb{Z}\}.$$

This is strictly larger than $\mathbb{Z}[\sqrt{5}]$!

(5) $\mathcal{O}_{\mathbb{Q}(\omega)} = \mathbb{Z}[\omega]$: The *Eisenstein integers*, where $\omega = e^{2\pi i/3}$. These form a hexagonal lattice in $\mathbb{C}$.

The general pattern: for $\mathbb{Q}(\sqrt{d})$ with d square-free,

$$\mathcal{O}_{\mathbb{Q}(\sqrt{d})} = \begin{cases} \mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right] & \text{if } d \equiv 1 \pmod 4 \\ \mathbb{Z}[\sqrt{d}] & \text{if } d \equiv 2, 3 \pmod 4 \end{cases}$$

Integral Basis and Discriminant

Definition

Definition 24.3 (Integral Basis). An *integral basis* for $\mathcal{O}_K$ is a $\mathbb{Z}$-basis $\{\omega_1, \dots, \omega_n\}$ such that

$$\mathcal{O}_K = \mathbb{Z}\omega_1 \oplus \cdots \oplus \mathbb{Z}\omega_n = \{a_1\omega_1 + \cdots + a_n\omega_n : a_i \in \mathbb{Z}\}.$$

For $\mathbb{Q}(i)$, an integral basis is $\{1, i\}$. For $\mathbb{Q}(\sqrt{5})$, an integral basis is $\left\{1, \frac{1+\sqrt{5}}{2}\right\}$.

Definition

Definition 24.4 (Discriminant). Let $\{\omega_1, \dots, \omega_n\}$ be an integral basis for $\mathcal{O}_K$. The *discriminant* is

$$\Delta_K = \det\left(\operatorname{Tr}_{K/\mathbb{Q}}(\omega_i\omega_j)\right)^2.$$

This is independent of the choice of integral basis and is an important invariant of the number field.

Example

Example 24.4 (Discriminants).

- $\Delta_{\mathbb{Q}(i)} = -4$
- $\Delta_{\mathbb{Q}(\sqrt{d})} = d$ if $d \equiv 1 \pmod 4$, and $4d$ if $d \equiv 2, 3 \pmod 4$
- $\Delta_{\mathbb{Q}(\sqrt{-5})} = -20$

Definition

Definition 24.5 (Norm and Trace). Let $K/\mathbb{Q}$ be a number field of degree n, and let $\alpha \in K$. Let $\sigma_1, \ldots, \sigma_n : K \to \mathbb{C}$ be the n embeddings of K into $\mathbb{C}$ (extending the identity on $\mathbb{Q}$). Define:

Norm:

$$N_{K/\mathbb{Q}}(\alpha) = \prod_{i=1}^{n} \sigma_i(\alpha)$$

(the product of all conjugates).

Trace:

$$\mathrm{Tr}_{K/\mathbb{Q}}(\alpha) = \sum_{i=1}^{n} \sigma_i(\alpha)$$

(the sum of all conjugates).

Both $N(\alpha)$ and $\mathrm{Tr}(\alpha)$ lie in $\mathbb{Q}$. If $\alpha \in \mathcal{O}_K$, then $N(\alpha), \mathrm{Tr}(\alpha) \in \mathbb{Z}$.

Example

Example 24.5 (Computing Norms and Traces).

(1) For $\alpha = a + bi \in \mathbb{Q}(i)$: The two embeddings are $\sigma_1(\alpha) = a + bi$ and $\sigma_2(\alpha) = a - bi$. Thus:

$$N(\alpha) = (a + bi)(a - bi) = a^2 + b^2, \quad \mathrm{Tr}(\alpha) = (a + bi) + (a - bi) = 2a.$$

(2) For $\alpha = a + b\sqrt{d} \in \mathbb{Q}(\sqrt{d})$:

$$N(\alpha) = (a + b\sqrt{d})(a - b\sqrt{d}) = a^2 - db^2, \quad \mathrm{Tr}(\alpha) = 2a.$$

(3) For $\alpha = \sqrt[3]{2} \in \mathbb{Q}(\sqrt[3]{2})$: The three conjugates are $\sqrt[3]{2}$, $\omega\sqrt[3]{2}$, $\omega^2\sqrt[3]{2}$ where $\omega = e^{2\pi i/3}$. Thus:

$$N(\alpha) = \sqrt[3]{2} \cdot \omega\sqrt[3]{2} \cdot \omega^2\sqrt[3]{2} = 2 \cdot (1 \cdot \omega \cdot \omega^2) = 2.$$

Theorem

Theorem 24.2 (Properties of Norm).

- $N(\alpha\beta) = N(\alpha)N(\beta)$ (multiplicative)
- $N(\alpha) = 0$ iff $\alpha = 0$
- If $\alpha \in \mathcal{O}_K$ and $N(\alpha) = \pm 1$, then α is a unit in $\mathcal{O}_K$

Proof. Multiplicativity follows from $N(\alpha\beta) = \prod \sigma_i(\alpha\beta) = \prod \sigma_i(\alpha)\sigma_i(\beta) = N(\alpha)N(\beta)$. If $N(\alpha) = \pm 1$, then $\alpha \cdot \frac{N(\alpha)}{\alpha} = N(\alpha) = \pm 1$, and $\frac{N(\alpha)}{\alpha} = \prod_{i \neq 1} \sigma_i(\alpha) \in \mathcal{O}_K$. Thus α has a multiplicative inverse in $\mathcal{O}_K$. ∎

Example

Example 24.6 (Units in Gaussian Integers). The units in $\mathbb{Z}[i]$ are those $\alpha = a + bi$ with $N(\alpha) = a^2 + b^2 = 1$. The only integer solutions are $(\pm 1, 0)$ and $(0, \pm 1)$, giving units:

$$\mathbb{Z}[i]^* = \{\pm 1, \pm i\}.$$

These are the four 4th roots of unity.

24.2 — Unique Factorization and Ideal Theory

One of the most shocking discoveries in 19th century mathematics was that unique factorization can fail in rings of algebraic integers. This crisis led to the development of ideal theory—one of the triumphs of abstract algebra.

The Failure of Unique Factorization

Example

Example 24.7 (Factorization in $\mathbb{Z}[\sqrt{-5}]$). Consider the ring $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\}$. We have:

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5}).$$

We claim these are *two different factorizations* into irreducibles. First, let's check the second factorization:

$$(1 + \sqrt{-5})(1 - \sqrt{-5}) = 1 - (-5) = 6. \quad \checkmark$$

Claim: The elements $2, 3, 1 + \sqrt{-5}, 1 - \sqrt{-5}$ are all irreducible in $\mathbb{Z}[\sqrt{-5}]$.

Proof that 2 is irreducible: Suppose $2 = \alpha\beta$ with $\alpha, \beta \in \mathbb{Z}[\sqrt{-5}]$, neither a unit. Taking norms:

$$4 = N(2) = N(\alpha)N(\beta).$$

For $\alpha = a + b\sqrt{-5}$, we have $N(\alpha) = a^2 + 5b^2$. The possible values with $N(\alpha) > 1$ and $N(\alpha)|4$ are $N(\alpha) \in \{2, 4\}$.

If $N(\alpha) = 2$, then $a^2 + 5b^2 = 2$. But checking: $b = 0$ gives $a^2 = 2$ (no integer solution), and $b = \pm 1$ gives $a^2 = -3$ (impossible). So $N(\alpha) \neq 2$.

Thus $N(\alpha) = 4$ or $N(\alpha) = 1$. If $N(\alpha) = 1$, then α is a unit (contradiction). If $N(\alpha) = 4$, then $N(\beta) = 1$, so β is a unit. Therefore 2 is irreducible.

Similar arguments show 3, $1 \pm \sqrt{-5}$ are irreducible.

Claim: These factorizations are genuinely different—no irreducible from one is an associate of an irreducible from the other.

If $2 = u(1 + \sqrt{-5})$ for a unit u, taking norms: $4 = N(u) \cdot 6$, giving $N(u) = 2/3$ (impossible for a unit). Similar arguments for other pairs. Thus the factorizations are distinct.

This failure of unique factorization was deeply troubling to 19th century mathematicians. Kummer attempted to "restore" unique factorization by

introducing "ideal numbers"—hypothetical elements that would make factorization unique. Dedekind made this rigorous through the theory of ideals.

Ideals and Dedekind Domains

> **Definition**
>
> **Definition 24.6 (Dedekind Domain).** An integral domain R is a *Dedekind domain* if:
>
> - R is Noetherian (every ideal is finitely generated)
> - R is integrally closed in its field of fractions
> - Every nonzero prime ideal of R is maximal

> **Theorem**
>
> **Theorem 24.3.** The ring of integers $\mathcal{O}_K$ of any number field K is a Dedekind domain.

> **Theorem**
>
> **Theorem 24.4 (Unique Factorization of Ideals).** Let R be a Dedekind domain. Every nonzero ideal $I \subseteq R$ factors uniquely as a product of prime ideals:
>
> $$I = \mathfrak{p}_1^{e_1}\mathfrak{p}_2^{e_2}\cdots\mathfrak{p}_k^{e_k}$$
>
> where $\mathfrak{p}_i$ are distinct prime ideals and $e_i \geq 1$. The $\mathfrak{p}_i$ and e_i are unique (up to ordering).

This is the fundamental theorem of ideal theory—unique factorization holds not for elements but for ideals! In Dedekind domains, ideals are the "correct" objects

to consider.

Example

Example 24.8 (Ideal Factorization in $\mathbb{Z}[\sqrt{-5}]$). Consider the principal ideal (6). We have:

$$(6) = (2, 1+\sqrt{-5})^2 (3, 1+\sqrt{-5})(3, 1-\sqrt{-5}).$$

The ideals $\mathfrak{p}_2 = (2, 1+\sqrt{-5})$, $\mathfrak{p}_3 = (3, 1+\sqrt{-5})$, $\mathfrak{p}_3' = (3, 1-\sqrt{-5})$ are prime ideals. Note:

$$\mathfrak{p}_2^2 = (2), \quad \mathfrak{p}_3 \mathfrak{p}_3' = (3), \quad \mathfrak{p}_2^2 \mathfrak{p}_3 \mathfrak{p}_3' = (6).$$

The two factorizations of the element 6 correspond to different ways of grouping these prime ideals:

$$(2)(3) = \mathfrak{p}_2^2 \cdot \mathfrak{p}_3 \mathfrak{p}_3', \quad (1+\sqrt{-5})(1-\sqrt{-5}) = \mathfrak{p}_2 \mathfrak{p}_3 \cdot \mathfrak{p}_2 \mathfrak{p}_3'.$$

At the level of ideals, there is no ambiguity—unique factorization is restored!

The Ideal Class Group

Definition

Definition 24.7 (Fractional Ideal, Ideal Class Group). A *fractional ideal* of $\mathcal{O}_K$ is a finitely generated $\mathcal{O}_K$-submodule of K. Two fractional ideals I and J are *equivalent* if there exist nonzero $\alpha, \beta \in \mathcal{O}_K$ with $\alpha I = \beta J$.

The *ideal class group* $\mathrm{Cl}(K)$ is the group of fractional ideals modulo this equivalence relation (with multiplication of ideals). The *class number* is

$$h_K = |\mathrm{Cl}(K)|.$$

Theorem

Theorem 24.5. $\mathcal{O}_K$ has unique factorization of elements (is a UFD) if and only if $h_K = 1$.

Example

Example 24.9 (Class Numbers).

- $h_{\mathbb{Q}(i)} = 1$ — Gaussian integers have UFD
- $h_{\mathbb{Q}(\sqrt{-5})} = 2$ — explains the failure of UFD
- $h_{\mathbb{Q}(\sqrt{-23})} = 3$
- $h_{\mathbb{Q}(\sqrt{-163})} = 1$ — the largest imaginary quadratic field with class number 1

The class number measures "how far" $\mathcal{O}_K$ is from being a UFD. Computing class numbers is a deep and difficult problem in algebraic number theory.

☯ Zen Meditation ☯

Ideals: When Elements Fail, Structure Prevails

The river does not factor uniquely into drops—
yet the flow patterns persist.
When elements fail to capture the essence,
we rise to a higher level: ideals.

Not the thing itself, but the principle it embodies.
Not the particular, but the universal.
At this level, order is restored.
Chaos was only an illusion of insufficient abstraction.

24.3 — Ramification Theory

When we extend from $\mathbb{Q}$ to a number field K, how do prime numbers behave? This question—central to algebraic number theory—is answered by ramification theory.

Decomposition of Primes in Extensions

> **Theorem**
>
> **Theorem 24.6 (Prime Decomposition).** Let K be a number field of degree $n = [K : \mathbb{Q}]$, and let p be a prime in $\mathbb{Z}$. The ideal $p\mathcal{O}_K$ factors as:
>
> $$p\mathcal{O}_K = \mathfrak{P}_1^{e_1}\mathfrak{P}_2^{e_2}\cdots\mathfrak{P}_g^{e_g}$$
>
> where $\mathfrak{P}_i$ are distinct prime ideals of $\mathcal{O}_K$ lying over p, and the integers e_i (ramification indices) and $f_i = [\mathcal{O}_K/\mathfrak{P}_i : \mathbb{Z}/p\mathbb{Z}]$ (inertial degrees) satisfy:
>
> $$\sum_{i=1}^{g} e_i f_i = n.$$

We say:

- p *ramifies* if some $e_i > 1$
- p is *unramified* if all $e_i = 1$
- p *splits completely* if all $e_i = f_i = 1$ (so $g = n$)
- p is *inert* if $g = 1$, $e_1 = 1$, $f_1 = n$

Example

Example 24.10 (Primes in $\mathbb{Q}(i)$). For $\mathbb{Q}(i)/\mathbb{Q}$ with $n = 2$, a prime $p \in \mathbb{Z}$ behaves as follows:

(1) $p = 2$ ramifies:

$$(2) = (1+i)^2(1-i)^2 = ((1+i)(1-i))^2 = (2 \cdot 1)^2 = \mathfrak{P}^2$$

where $\mathfrak{P} = (1+i)$. We have $e = 2, f = 1, g = 1$.

(2) $p \equiv 1 \pmod 4$ splits completely: For example, $5 = (2+i)(2-i)$. Both factors are prime in $\mathbb{Z}[i]$. We have $g = 2, e_1 = e_2 = 1, f_1 = f_2 = 1$.

(3) $p \equiv 3 \pmod 4$ remains inert: For example, 3 remains prime in $\mathbb{Z}[i]$. We have $g = 1, e = 1, f = 2$.

This is intimately connected to Fermat's theorem on sums of two squares: $p = a^2 + b^2$ has integer solutions iff $p = 2$ or $p \equiv 1 \pmod 4$, which is exactly when p is not inert in $\mathbb{Z}[i]$!

Example

Example 24.11 (Quadratic Fields $\mathbb{Q}(\sqrt{d})$). For $\mathbb{Q}(\sqrt{d})$ with d square-free:

(1) Primes dividing the discriminant ramify.

(2) Other primes either split completely or remain inert, determined by whether d is a quadratic residue mod p:

- p splits if $\left(\frac{d}{p}\right) = 1$ (Legendre symbol)
- p is inert if $\left(\frac{d}{p}\right) = -1$

The Frobenius Element

For Galois extensions, there is a beautiful connection between prime decomposition and the Galois group.

Definition

Definition 24.8 (Frobenius Element). Let $K/\mathbb{Q}$ be a Galois extension with Galois group G, and let $\mathfrak{P}$ be a prime of $\mathcal{O}_K$ lying over p, unramified. The *Frobenius element* $\mathrm{Frob}_{\mathfrak{P}} \in G$ is the unique element satisfying:

$$\mathrm{Frob}_{\mathfrak{P}}(\alpha) \equiv \alpha^p \pmod{\mathfrak{P}}$$

for all $\alpha \in \mathcal{O}_K$.

The Frobenius element encodes how the prime p splits: the cycle structure of $\mathrm{Frob}_{\mathfrak{P}}$ determines the factorization pattern!

Theorem

Theorem 24.7 (Chebotarev Density Theorem). Let $K/\mathbb{Q}$ be a Galois extension with Galois group G, and let C be a conjugacy class in G. The set of primes p (unramified) such that $\mathrm{Frob}_p \in C$ has density

$$\frac{|C|}{|G|}.$$

This remarkable theorem generalizes Dirichlet's theorem on primes in arithmetic progressions and is fundamental to modern number theory.

24.4 — Class Field Theory

Class field theory, developed by Kronecker, Weber, Hilbert, Takagi, and Artin, provides a stunning connection between Galois theory and ideal theory.

Abelian Extensions of $\mathbb{Q}$

Theorem

Theorem 24.8 (Kronecker-Weber Theorem). Every finite abelian extension of $\mathbb{Q}$ is contained in a cyclotomic field $\mathbb{Q}(\zeta_n)$ for some n.

In other words, if $K/\mathbb{Q}$ is Galois with abelian Galois group, then $K \subseteq \mathbb{Q}(\zeta_n)$ for some n. This completely classifies abelian extensions of $\mathbb{Q}$!

Example

Example 24.12.

- $\mathbb{Q}(i) \subseteq \mathbb{Q}(\zeta_4)$ since $i = \zeta_4$
- $\mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\zeta_8)$ since $\sqrt{2} = \zeta_8 + \zeta_8^{-1}$
- $\mathbb{Q}(\sqrt{5}) \subseteq \mathbb{Q}(\zeta_5)$ since $\sqrt{5} = \zeta_5 + \zeta_5^4 - \zeta_5^2 - \zeta_5^3$

Artin Reciprocity

Theorem

Theorem 24.9 (Artin Reciprocity). For an abelian extension $K/\mathbb{Q}$, there is a canonical isomorphism (the *Artin map*):

$$\theta : I_K / P_K \to \mathrm{Gal}(K/\mathbb{Q})$$

where I_K is the group of fractional ideals coprime to ramified primes, and P_K is the subgroup of principal ideals.

This theorem generalizes quadratic reciprocity and connects the arithmetic of ideals to the Galois group. It is one of the deepest results in all of mathematics.

24.5 — Fermat's Last Theorem

We conclude with one of the most famous problems in mathematics, whose resolution required the full power of algebraic number theory and far beyond.

> **Theorem**
>
> **Theorem 24.10 (Fermat's Last Theorem).** For any integer $n \geq 3$, the equation
>
> $$x^n + y^n = z^n$$
>
> has no non-trivial integer solutions (i.e., no solutions with $xyz \neq 0$).

Historical Context

In 1637, Pierre de Fermat wrote in the margin of his copy of Diophantus' *Arithmetica*: "I have discovered a truly marvelous proof of this proposition, which this margin is too narrow to contain." For over 350 years, this claim tantalized mathematicians.

Fermat himself proved the case $n = 4$ using infinite descent. Euler proved $n = 3$ (with a gap later filled). But general progress stalled until the 19th century development of algebraic number theory.

Kummer's Approach

Ernst Kummer made dramatic progress using cyclotomic fields. The key idea: factor $x^p + y^p$ in $\mathbb{Q}(\zeta_p)$ where $\zeta_p = e^{2\pi i/p}$ and p is an odd prime:

$$x^p + y^p = \prod_{j=0}^{p-1} (x + \zeta_p^j y).$$

If $x^p + y^p = z^p$, these factors have a special structure. Kummer showed that if $\mathbb{Z}[\zeta_p]$ had unique factorization, Fermat's Last Theorem would hold for exponent p.

But unique factorization often fails! Kummer introduced "ideal numbers" (precursors to ideals) to handle this. He defined:

Definition

Definition 24.9 (Regular Prime). An odd prime p is *regular* if p does not divide the class number $h_{\mathbb{Q}(\zeta_p)}$.

Theorem

Theorem 24.11 (Kummer). Fermat's Last Theorem holds for all regular primes.

The first irregular prime is 37, and Kummer proved FLT for all primes less than 100. But infinitely many irregular primes exist, so this approach could not finish the proof.

The Modern Proof: Wiles (1995)

Andrew Wiles' proof, completed in 1995 with Richard Taylor, took a completely different approach via *elliptic curves*.

Definition

Definition 24.10 (Elliptic Curve). An *elliptic curve* over $\mathbb{Q}$ is a smooth projective curve of genus 1 with a rational point, given by a Weierstrass equation:

$$y^2 = x^3 + ax + b, \quad a, b \in \mathbb{Q}, \quad 4a^3 + 27b^2 \neq 0.$$

The Taniyama-Shimura-Weil Conjecture

> **Theorem**
>
> **Conjecture (Taniyama-Shimura-Weil, now Theorem).** Every elliptic curve over $\mathbb{Q}$ is *modular*—it arises from a modular form.

This deep connection between elliptic curves (algebraic geometry) and modular forms (complex analysis) was completely unexpected.

The Frey Curve

In 1985, Gerhard Frey made a crucial observation: if $a^p + b^p = c^p$ were a solution to Fermat's equation, the elliptic curve

$$y^2 = x(x - a^p)(x + b^p)$$

would have strange properties. In 1986, Ken Ribet proved:

> **Theorem**
>
> **Theorem 24.12 (Ribet, 1986).** The Frey curve associated to a hypothetical FLT solution cannot be modular.

This reduced FLT to the Taniyama-Shimura-Weil conjecture! If all elliptic curves are modular, the Frey curve cannot exist, so FLT holds.

Wiles' Proof

Wiles proved the Taniyama-Shimura-Weil conjecture for *semistable* elliptic curves—precisely those including the Frey curve. His proof used:

- Galois representations
- Modular forms and Hecke algebras
- Deformation theory of Galois representations
- Iwasawa theory
- Commutative algebra (Gorenstein rings)

The proof spans over 100 pages of extremely sophisticated mathematics. A gap discovered shortly after the initial announcement was fixed by Wiles and Taylor in a subsequent paper.

> ⚡ **Greek Meditation** ⚡
>
> **The Pythagorean Dream: Number as Reality**
>
> "All is number," proclaimed Pythagoras.
> From this seed grew number theory—
> the study of the integers and their extensions.
>
> Fermat's marginal note sparked a 350-year quest,
> driving the development of entire mathematical fields.
> Algebraic number theory, elliptic curves, modular forms—
> all tools forged in pursuit of this single question.
>
> When Wiles completed the proof in 1995,
> it was not just one theorem proven,
> but a demonstration of the unity of mathematics itself.
> The question was arithmetic; the answer required
> algebra, geometry, analysis, all intertwined.
>
> The Pythagoreans were right: number is essence.
> But its essence reveals itself only through
> the totality of mathematical structure.

Practice Problems

Problem 24.1. Show that $\sqrt{2} + \sqrt{3}$ is an algebraic integer.

> ***Solution.*** Let $\alpha = \sqrt{2} + \sqrt{3}$. Then:

$$\alpha^2 = 2 + 2\sqrt{6} + 3 = 5 + 2\sqrt{6}.$$

So $\alpha^2 - 5 = 2\sqrt{6}$, and $(\alpha^2 - 5)^2 = 24$. Thus:

$$\alpha^4 - 10\alpha^2 + 25 = 24 \implies \alpha^4 - 10\alpha^2 + 1 = 0.$$

This is monic with integer coefficients, so α is an algebraic integer.

Problem 24.2. Compute the norm and trace of $3 + 4i$ in $\mathbb{Q}(i)$.

Solution. The conjugates of $3 + 4i$ are $3 + 4i$ and $3 - 4i$. Thus:

$$N(3 + 4i) = (3 + 4i)(3 - 4i) = 9 + 16 = 25.$$

$$\mathrm{Tr}(3 + 4i) = (3 + 4i) + (3 - 4i) = 6.$$

Problem 24.3. Find all units in $\mathbb{Z}[\sqrt{-2}]$.

Solution. Units $\alpha = a + b\sqrt{-2}$ satisfy $N(\alpha) = a^2 + 2b^2 = 1$. Since $a, b \in \mathbb{Z}$, we need $a^2 + 2b^2 = 1$.

If $b \neq 0$, then $2b^2 \geq 2$, so $a^2 + 2b^2 \geq 2 > 1$. Thus $b = 0$, giving $a^2 = 1$, so $a = \pm 1$.

Therefore the units are $\mathbb{Z}[\sqrt{-2}]^* = \{\pm 1\}$.

Problem 24.4. Show that 2 is irreducible but not prime in $\mathbb{Z}[\sqrt{-5}]$.

Solution. We showed 2 is irreducible in Example 24.7 (using norms).

To show 2 is not prime: note that $2 | 6 = (1 + \sqrt{-5})(1 - \sqrt{-5})$, so 2 divides the product. But 2 does not divide either factor (if $2 | (1 + \sqrt{-5})$, then $1 + \sqrt{-5} = 2(a + b\sqrt{-5})$ for some $a, b \in \mathbb{Z}$, giving $1 = 2a$ and $1 = 2b$, impossible).

Thus 2 is irreducible but not prime, showing $\mathbb{Z}[\sqrt{-5}]$ is not a UFD.

Problem 24.5. Verify that in $\mathbb{Z}[i]$, the prime 5 factors as $5 = (2 + i)(2 - i)$.

Solution. Compute:

$$(2 + i)(2 - i) = 4 - i^2 = 4 + 1 = 5. \quad \checkmark$$

Both $2+i$ and $2-i$ are irreducible in $\mathbb{Z}[i]$: their norms are $N(2+i)=5$, which is prime in $\mathbb{Z}$. If $2+i=\alpha\beta$, then $5=N(\alpha)N(\beta)$, forcing one factor to have norm 1 (be a unit).

Problem 24.6. Determine how the prime 7 factors in $\mathbb{Q}(i)$.

Solution. We check if -1 is a quadratic residue mod 7. By Euler's criterion:

$$(-1)^{(7-1)/2} = (-1)^3 = -1 \not\equiv 1 \pmod{7}.$$

So -1 is not a quadratic residue mod 7, meaning $x^2 \equiv -1 \pmod{7}$ has no solutions.

Therefore 7 remains inert in $\mathbb{Q}(i)$: the ideal (7) is prime in $\mathbb{Z}[i]$.

Problem 24.7. Show that $\mathbb{Z}[\sqrt{-1}]$ and $\mathbb{Z}[i]$ are the same ring.

Solution. Both denote $\{a+b\sqrt{-1} : a,b\in\mathbb{Z}\} = \{a+bi : a,b\in\mathbb{Z}\}$ since $i=\sqrt{-1}$. They are identical.

Problem 24.8. Prove that if $\alpha \in \mathcal{O}_K$ and $N(\alpha)=\pm 1$, then α is a unit.

Solution. We have $N(\alpha)=\prod_{i=1}^{n}\sigma_i(\alpha)$ where σ_i are the embeddings $K\to\mathbb{C}$. If $N(\alpha)=\pm1$, then

$$\alpha\cdot\frac{\pm 1}{\alpha}=\pm 1.$$

Note that $\beta=\frac{N(\alpha)}{\alpha}=\frac{\pm1}{\alpha}=\pm\prod_{i=2}^{n}\sigma_i(\alpha)$. Since each $\sigma_i(\alpha)$ is an algebraic integer (apply σ_i to the minimal polynomial), $\beta\in\mathcal{O}_K$.

Thus $\alpha\beta=\pm1$, showing α has a multiplicative inverse in $\mathcal{O}_K$.

Problem 24.9. Find the discriminant of $\mathbb{Q}(\sqrt{-5})$.

Solution. Since $-5\equiv 3 \pmod{4}$, we have $\mathcal{O}_{\mathbb{Q}(\sqrt{-5})}=\mathbb{Z}[\sqrt{-5}]$ with integral basis $\{1,\sqrt{-5}\}$.

The discriminant formula for $\mathbb{Q}(\sqrt{d})$ with $d \equiv 2, 3 \pmod 4$ is $\Delta = 4d$. Thus:

$$\Delta_{\mathbb{Q}(\sqrt{-5})} = 4 \cdot (-5) = -20.$$

Problem 24.10. Determine the class number h_K for $K = \mathbb{Q}(i)$.

Solution. The class number is 1 because $\mathbb{Z}[i]$ is a UFD (every ideal is principal). Thus $h_{\mathbb{Q}(i)} = 1$.

Problem 24.11. Show that the equation $x^2 + 5 = y^3$ has no integer solutions with x odd.

Solution. Factor in $\mathbb{Z}[\sqrt{-5}]$:

$$y^3 = x^2 + 5 = (x + \sqrt{-5})(x - \sqrt{-5}).$$

The ideals $(x + \sqrt{-5})$ and $(x - \sqrt{-5})$ are coprime (their sum contains 1). In a UFD, coprime elements whose product is a cube must each be cubes (up to units). But $\mathbb{Z}[\sqrt{-5}]$ is not a UFD.

However, $h_{\mathbb{Q}(\sqrt{-5})} = 2$, so cubing ideals makes them principal. One can show this forces x even, contradicting the assumption.

Problem 24.12. For $K = \mathbb{Q}(\sqrt{2})$, determine how the prime 3 decomposes in $\mathcal{O}_K$.

Solution. We check if 2 is a quadratic residue mod 3. Compute $2^{(3-1)/2} = 2^1 = 2 \equiv -1 \pmod 3$. So 2 is not a QR mod 3.

Therefore 3 remains inert in $\mathbb{Q}(\sqrt{2})$: the ideal (3) is prime in $\mathcal{O}_K$.

Problem 24.13. Show that $\omega = e^{2\pi i/3}$ satisfies $\omega^2 + \omega + 1 = 0$.

Solution. We have $\omega^3 = 1$, so $\omega^3 - 1 = 0$. Factor:

$$\omega^3 - 1 = (\omega - 1)(\omega^2 + \omega + 1) = 0.$$

Since $\omega \neq 1$, we must have $\omega^2 + \omega + 1 = 0$.

Problem 24.14. Prove that $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ are not isomorphic as fields.

Solution. If they were isomorphic via $\phi : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{3})$, then $\phi(\sqrt{2})$ would satisfy $x^2 - 2 = 0$ (since ϕ preserves polynomial

equations). So $\phi(\sqrt{2})^2 = 2$.

But $\phi(\sqrt{2}) = a + b\sqrt{3}$ for some $a, b \in \mathbb{Q}$. Then:

$$(a + b\sqrt{3})^2 = a^2 + 3b^2 + 2ab\sqrt{3} = 2.$$

This requires $2ab = 0$ and $a^2 + 3b^2 = 2$.

If $a = 0$: $3b^2 = 2$, so $b^2 = 2/3$, impossible for $b \in \mathbb{Q}$.

If $b = 0$: $a^2 = 2$, so $a = \pm\sqrt{2} \notin \mathbb{Q}$.

Contradiction, so no such isomorphism exists.

Problem 24.15. Compute the norm of $\sqrt[3]{2}$ as an element of $\mathbb{Q}(\sqrt[3]{2})$ viewed as an extension of $\mathbb{Q}$.

Solution. The minimal polynomial of $\sqrt[3]{2}$ over $\mathbb{Q}$ is $x^3 - 2$. The three roots (conjugates) are $\sqrt[3]{2}, \omega\sqrt[3]{2}, \omega^2\sqrt[3]{2}$ where $\omega = e^{2\pi i/3}$.

The norm is:

$$N(\sqrt[3]{2}) = \sqrt[3]{2} \cdot \omega\sqrt[3]{2} \cdot \omega^2\sqrt[3]{2} = 2 \cdot \omega \cdot \omega^2 = 2 \cdot \omega^3 = 2 \cdot 1 = 2.$$

Problem 24.16. Why does Kummer's method not immediately prove Fermat's Last Theorem for all exponents?

Solution. Kummer's method works when $\mathbb{Z}[\zeta_p]$ has unique factorization, or more generally when $p \nmid h_{\mathbb{Q}(\zeta_p)}$ (regular primes). But not all primes are regular—the first irregular prime is 37.

While Kummer proved FLT for all regular primes, there are infinitely many irregular primes, so this approach alone cannot finish the proof. Additional techniques are needed to handle irregular primes.

Problem 24.17. Show that $\mathcal{O}_{\mathbb{Q}(\sqrt{5})} \neq \mathbb{Z}[\sqrt{5}]$.

Solution. Since $5 \equiv 1 \pmod 4$, we have

$$\mathcal{O}_{\mathbb{Q}(\sqrt{5})} = \mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right].$$

Note that $\phi = \frac{1+\sqrt{5}}{2}$ is an algebraic integer (satisfies $x^2 - x - 1 = 0$) but $\phi \notin \mathbb{Z}[\sqrt{5}]$ (would require writing $\frac{1+\sqrt{5}}{2} = a + b\sqrt{5}$ with $a, b \in \mathbb{Z}$, giving $a = 1/2$, impossible).

Thus $\mathbb{Z}[\sqrt{5}]$ is a proper subring of $\mathcal{O}_{\mathbb{Q}(\sqrt{5})}$.

Problem 24.18. Explain the connection between Fermat's theorem on sums of two squares and prime splitting in $\mathbb{Z}[i]$.

Solution. Fermat's theorem: An odd prime p can be written as $p = a^2 + b^2$ iff $p \equiv 1 \pmod 4$.

In $\mathbb{Z}[i]$, we have $N(a + bi) = a^2 + b^2$. So $p = a^2 + b^2$ iff $p = N(a + bi) = (a + bi)(a - bi)$, i.e., p factors in $\mathbb{Z}[i]$.

From ramification theory: p splits in $\mathbb{Q}(i)$ iff $\left(\frac{-1}{p}\right) = 1$ iff $p \equiv 1 \pmod 4$.

Thus Fermat's theorem is equivalent to the statement that odd primes $p \equiv 1 \pmod 4$ split in $\mathbb{Z}[i]$!

Problem 24.19. State the fundamental identity $\sum e_i f_i = n$ for prime decomposition and explain its significance.

Solution. When prime $p \in \mathbb{Z}$ decomposes in $\mathcal{O}_K$ (with $[K : \mathbb{Q}] = n$) as

$$p\mathcal{O}_K = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_g^{e_g},$$

we have $\sum_{i=1}^{g} e_i f_i = n$ where e_i is the ramification index and $f_i = [\mathcal{O}_K/\mathfrak{P}_i : \mathbb{Z}/p\mathbb{Z}]$ is the inertial degree.

Significance: This identity constrains how primes can decompose. For example, in a quadratic field ($n = 2$):

- Inert: $g = 1, e = 1, f = 2$ (one prime, stays prime)
- Split: $g = 2, e_1 = e_2 = 1, f_1 = f_2 = 1$ (two primes)
- Ramified: $g = 1, e = 2, f = 1$ (one prime, squared)

Problem 24.20. (Challenge) Outline why unique factorization of ideals holds in Dedekind domains.

Solution. (Outline) Key steps:

Existence: Use Noetherian property: every ideal contains a product of prime ideals. Choose a minimal such product—this must be the factorization.

Uniqueness: Suppose $\mathfrak{p}_1 \cdots \mathfrak{p}_k = \mathfrak{q}_1 \cdots \mathfrak{q}_\ell$. Since $\mathfrak{p}_1$ is prime and divides the product, $\mathfrak{p}_1 \supseteq \mathfrak{q}_j$ for some j. But prime ideals are maximal in Dedekind domains, so $\mathfrak{p}_1 = \mathfrak{q}_j$. Cancel and repeat.

☯ Zen Meditation ☯

Primes Splitting: One Becoming Many, Many Returning to One

A single prime, complete in $\mathbb{Z}$,
enters the number field and splits—
one becomes two, or three, or remains one.

Like light through a prism,
the prime reveals its hidden structure.
In the splitting is information;
in the ramification, memory of origin.

The extension does not destroy the prime—
it reveals what was always there,
latent, waiting to unfold.

Many primes above, one prime below.
One prime below, many primes above.
The dance of unity and diversity,
forever intertwined.

"In mathematics, you don't understand things.
You just get used to them."
— John von Neumann

And yet, in getting used to them,
we discover beauty beyond imagination—
primes that split and ramify,
ideals that restore order,
and theorems that connect
the arithmetic of today
with the dreams of antiquity.

⚘ End of Chapter 24 ⚘

Chapter 25: Modern Applications

"Symmetry is one idea by which man through the ages
has tried to comprehend and create order, beauty, and perfection."
— Hermann Weyl

The journey begun five volumes ago now reaches its culmination.
From humble beginnings with groups and rings,
through fields and Galois theory,
to the heights of modern abstraction—
we witness abstract algebra alive in the world,
powering technology, revealing nature's secrets,
and continuing to evolve.

Chapter Overview:

- **25.1** — Cryptography: Securing the Digital Age
- **25.2** — Coding Theory: Reliable Communication
- **25.3** — Computer Science Applications
- **25.4** — Physics and Chemistry
- **25.5** — The Journey Complete: Synthesis & Future

Chapter 25 — Modern Applications & Conclusion

Introduction: Abstract Algebra in the Modern World

When the pioneers of abstract algebra—Galois, Abel, Dedekind, Noether—developed their theories in the 19th and early 20th centuries, they were driven primarily by internal mathematical questions. How do we solve polynomial equations? Why does unique factorization sometimes fail? What is the true nature of number?

They could not have imagined that their abstract investigations would, within a century, become essential to the functioning of modern civilization. Today, every time you make an online purchase, stream a video, or send a text message, you rely on abstract algebra. The same mathematical structures that once seemed pure abstraction now protect your credit card, compress your data, and encode your communications.

But the applications go far deeper than technology. Abstract algebra reveals the fundamental symmetries of nature itself. From the behavior of subatomic particles to the

structure of crystals, from the geometry of spacetime to the architecture of molecules, algebraic structures provide the language in which nature speaks.

> ⚡ **Greek Meditation** ⚡
>
> **The Completion: Telos and the Fulfillment of Purpose**
>
> Aristotle taught that everything has a τέλος (telos)—
> an end, a purpose, a completion toward which it strives.
> The acorn's telos is the mighty oak.
> The student's telos is wisdom.
>
> What is the telos of abstract algebra?
> Not merely to solve equations or classify structures,
> but to reveal the deep unity underlying all mathematics—
> and through mathematics, all of reality.
>
> In this final chapter, we witness the fulfillment:
> theory meeting practice, abstraction becoming concrete,
> the circle completing itself.
> Form and Function unite at last.

In this culminating chapter, we explore how abstract algebra powers the modern world—from the cryptography that secures the internet to the error-correcting codes that enable space exploration. We see how it shapes our understanding of quantum mechanics and guides the design of new materials. And we reflect on the remarkable journey through five volumes, synthesizing what we have learned and glimpsing what lies ahead.

The journey that began with basic group axioms now reveals itself as a unified exploration of structure, symmetry, and transformation. The techniques developed across 25 chapters form not just a collection of results, but a coherent worldview—a way of seeing mathematical reality.

25.1 — Cryptography: Securing the Digital Age

Modern cryptography—the science of secure communication—is fundamentally an application of abstract algebra. Nearly every cryptographic system in use today relies on

algebraic structures: groups, rings, fields, and their properties.

RSA Cryptosystem

The RSA algorithm, invented in 1977 by Rivest, Shamir, and Adleman, revolutionized cryptography by enabling *public-key* encryption. Its security rests on number theory and the difficulty of factoring large integers.

> **Theorem**
>
> **The RSA Algorithm.**
>
> **Key Generation:**
>
> 1. Choose two large primes p, q (typically 1024+ bits each)
> 2. Compute $n = pq$ and $\phi(n) = (p-1)(q-1)$
> 3. Choose e with $\gcd(e, \phi(n)) = 1$ (public exponent, often $e = 65537$)
> 4. Compute $d = e^{-1} \pmod{\phi(n)}$ (private exponent)
> 5. **Public key:** (n, e); **Private key:** (n, d)
>
> **Encryption:** To encrypt message $m < n$: $c \equiv m^e \pmod{n}$
>
> **Decryption:** To decrypt ciphertext c: $m \equiv c^d \pmod{n}$

> ***Proof.*** **Why it works:** By Euler's theorem (from group theory!), for m coprime to n :
>
> $$m^{\phi(n)} \equiv 1 \pmod{n}.$$
>
> Since $ed \equiv 1 \pmod{\phi(n)}$, we have $ed = 1 + k\phi(n)$ for some k. Thus:
>
> $$c^d \equiv (m^e)^d = m^{ed} = m^{1+k\phi(n)} = m \cdot (m^{\phi(n)})^k \equiv m \cdot 1^k = m \pmod{n}.$$
>
> ∎

Example

Example 25.1 (Toy RSA). Let $p = 61, q = 53$. Then $n = 3233$ and $\phi(n) = 60 \cdot 52 = 3120$.

Choose $e = 17$ (check: $\gcd(17, 3120) = 1$). Compute $d = 17^{-1} \pmod{3120} = 2753$.

Public key: $(3233, 17)$. Private key: $(3233, 2753)$.

Encrypt $m = 123$**:** $c \equiv 123^{17} \pmod{3233}$. Computing: $c = 855$.

Decrypt: $m \equiv 855^{2753} \pmod{3233}$. Computing: $m = 123$. ✓

Security: Breaking RSA requires factoring $n = pq$. For large n (2048+ bits), no efficient factoring algorithm is known. This is not proven impossible—factoring might be in P—but centuries of mathematical effort haven't found a fast method.

The algebraic foundations: Euler's theorem (consequence of Lagrange's theorem for $(\mathbb{Z}/n\mathbb{Z})^*$), Chinese Remainder Theorem (from ring theory), and properties of finite cyclic groups.

Elliptic Curve Cryptography (ECC)

Elliptic curves over finite fields provide even stronger cryptography with smaller key sizes.

Definition

Definition 25.1 (Elliptic Curve over Finite Field). An elliptic curve over $\mathbb{F}_p$ (prime $p > 3$) is defined by:

$$E : y^2 \equiv x^3 + ax + b \pmod{p}$$

with $4a^3 + 27b^2 \not\equiv 0 \pmod{p}$. The points $E(\mathbb{F}_p)$ form a group under a geometric addition law!

The group law makes elliptic curves a powerful tool for cryptography. The *Elliptic Curve Discrete Logarithm Problem* (ECDLP): given P and $Q = nP$, find n—appears much harder than the standard discrete log problem.

Example

Example 25.2 (ECC in Practice).

- **Bitcoin:** Uses elliptic curve secp256k1 for digital signatures
- **TLS/SSL:** Modern HTTPS connections use ECC for key exchange
- **Signal/WhatsApp:** End-to-end encryption via Curve25519

ECC achieves RSA-2048 equivalent security with only 224-bit keys—much more efficient!

Post-Quantum Cryptography

Quantum computers threaten RSA and ECC (Shor's algorithm can factor and compute discrete logs in polynomial time). The quest for *post-quantum* cryptography relies on different algebraic structures:

- **Lattice-based cryptography:** Uses lattices in $\mathbb{R}^n$, problems like SVP (shortest vector problem)
- **Code-based cryptography:** Based on error-correcting codes (see next section)
- **Multivariate cryptography:** Systems of polynomial equations over finite fields
- **Hash-based signatures:** Relies only on hash function security

Abstract algebra continues to evolve with threats and needs!

25.2 — Coding Theory: Reliable Communication

How do we transmit information reliably over noisy channels? Coding theory, born from Shannon's information theory and Hamming's pioneering work, provides the answer through *error-correcting codes*—deeply algebraic constructions.

Linear Codes

Definition

Definition 25.2 (Linear Code). An $[n, k]$ *linear code* over $\mathbb{F}_q$ is a k-dimensional subspace $C \subseteq \mathbb{F}_q^n$. We encode k-bit messages as n-bit codewords, with $n - k$ redundancy bits for error correction.

The *minimum distance* $d = \min\{d(c_1, c_2) : c_1 \neq c_2 \in C\}$ determines error-correcting capability: a code with minimum distance d can correct $\lfloor (d-1)/2 \rfloor$ errors.

Reed-Solomon Codes

Reed-Solomon codes are among the most important and widely used error-correcting codes, based on polynomial evaluation over finite fields.

Theorem

Construction 25.1 (Reed-Solomon Code). Fix a finite field $\mathbb{F}_q$ and distinct elements $\alpha_1, \ldots, \alpha_n \in \mathbb{F}_q$. The RS code $RS_k(n)$ consists of all vectors:

$$(f(\alpha_1), f(\alpha_2), \ldots, f(\alpha_n))$$

where $f(x) \in \mathbb{F}_q[x]$ has degree $< k$.

This gives an $[n, k, n-k+1]$ code (length n, dimension k, minimum distance $n - k + 1$).

Example

Example 25.3 (RS Codes in the Real World).

- **CDs and DVDs:** Reed-Solomon codes correct scratches and damage
- **QR codes:** Can be partially obscured and still scan correctly
- **Deep space communication:** Voyager probes use RS codes to transmit data from billions of miles away
- **Digital television (DVB):** Corrects transmission errors
- **RAID storage:** Recovers data from failed drives

The power of Reed-Solomon codes comes from algebra: evaluating polynomials (ring operations) and using properties of finite fields. Decoding uses the Euclidean algorithm —pure abstract algebra enabling modern technology!

BCH and Cyclic Codes

Cyclic codes are linear codes with additional structure: if $(c_0, c_1, \ldots, c_{n-1}) \in C$, then $(c_{n-1}, c_0, \ldots, c_{n-2}) \in C$ (cyclic shift). These correspond to ideals in the quotient ring $\mathbb{F}_q[x]/(x^n - 1)$!

BCH codes (Bose-Chaudhuri-Hocquenghem) generalize Reed-Solomon codes and are constructed using cyclotomic cosets and minimal polynomials—concepts from field theory and Galois theory.

25.3 — Computer Science Applications

Algebraic Algorithms

Abstract algebra isn't just used by computers—it's essential to *understanding* computation itself.

- **Computational algebra systems:** Mathematica, Maple, SageMath, GAP implement group theory, ring theory, Gröbner bases
- **Gröbner bases:** Algorithmic tool for solving systems of polynomial equations, with applications in robotics, optimization, automated theorem proving
- **Symbolic computation:** Computer algebra systems manipulate algebraic expressions exactly (not numerically)

Quantum Computing

Quantum computers operate on fundamentally different principles than classical computers—and group theory is essential to understanding them.

- **Quantum gates:** Unitary transformations (elements of $U(n)$, the unitary group)
- **Quantum error correction:** Stabilizer codes use subgroups of Pauli matrices
- **Shor's algorithm:** Uses quantum Fourier transform (representation theory of cyclic groups!) to factor integers

Machine Learning and Symmetry

Modern deep learning is discovering the importance of symmetry and group theory.

- **Convolutional neural networks (CNNs):** Exploit translational symmetry
- **Equivariant neural networks:** Designed to respect group symmetries, improving efficiency and generalization
- **Representation theory in ML:** Understanding how networks represent data uses concepts from representation theory

25.4 — Physics and Chemistry

Symmetry in Quantum Mechanics

> **Theorem**
>
> **Theorem 25.2 (Noether's Theorem).** Every continuous symmetry of a physical system corresponds to a conserved quantity:
>
> - Time translation symmetry → Energy conservation
> - Space translation symmetry → Momentum conservation
> - Rotational symmetry → Angular momentum conservation

This profound connection between symmetry (group theory) and conservation laws (physics) is one of the deepest insights in 20th century physics.

The Standard Model

The Standard Model of particle physics is fundamentally a theory of symmetry—specifically, a gauge theory with symmetry group:

$$SU(3) \times SU(2) \times U(1)$$

- $SU(3)$**:** Color charge (strong force). Quarks transform in the fundamental representation, gluons in the adjoint (8-dimensional)
- $SU(2)$**:** Weak isospin (weak force). Left-handed fermions form doublets
- $U(1)$**:** Hypercharge (electromagnetic force after symmetry breaking)

Representation theory of Lie groups determines:

- Which particles exist (irreducible representations)
- Which interactions are allowed (Clebsch-Gordan coefficients)
- Conservation laws and selection rules

Crystallography and Materials Science

Crystals are physical realizations of groups—their atomic structures are determined by the 230 *space groups* in three dimensions.

- **Point groups:** Rotational symmetries (32 crystallographic point groups)
- **Space groups:** Point groups plus translations (230 possibilities)
- **X-ray diffraction:** Fourier analysis (characters of representations!) reveals crystal structure
- **Materials design:** Understanding symmetry allows predicting and engineering material properties

> **☯ Zen Meditation ☯**
>
> **Applications: Theory Flowing Into Practice**
>
> The river flows from mountain to sea,
> never losing its nature as water.
> So too, abstract algebra flows
> from pure theory into application,
> never losing its essential structure.
>
> What seemed abstract becomes concrete.
> What seemed useless becomes indispensable.
> The distinction between pure and applied

> reveals itself as illusion—
> there is only mathematics,
> revealing itself in many forms.

25.5 — The Journey Complete: Synthesis and Future

"Mathematics is not about numbers, equations, computations, or algorithms:
it is about understanding."
— William Paul Thurston

The Arc of Five Volumes

We have traveled far together. Let us pause to survey the landscape we have traversed.

Volume I: Foundations

We began with groups—the mathematical abstraction of symmetry. From simple examples (integers under addition, permutations) we built a theory of structure-preserving maps (homomorphisms), quotient structures, and classification theorems. We learned that structure matters more than specific elements.

Key achievements: Lagrange's theorem, isomorphism theorems, group actions, Sylow theorems. We gained the language of modern algebra.

Volume II: Rings and Advanced Group Theory

We extended beyond single operations to rings—structures with both addition and multiplication. Polynomial rings, quotient rings, and ideals revealed deeper patterns. The Jordan-Hölder theorem showed that composition series provide canonical decompositions. Semidirect products allowed us to build complex groups from simpler pieces.

Key achievements: Ring theory foundations, polynomial arithmetic, unique factorization domains, classification of finite abelian groups, semidirect products.

Volume III: Fields and Extensions

Fields—where every nonzero element has a multiplicative inverse—enabled us to solve equations. Extension fields, algebraic vs transcendental elements, and constructions with straightedge and compass connected ancient problems to modern algebra. Finite fields opened doors to applications in coding theory and cryptography.

Key achievements: Field extensions, degree formulas, ruler and compass constructions, finite fields, splitting fields, separability.

Volume IV: Galois Theory

The crown jewel. Galois theory unified group theory and field theory, showing that the solvability of polynomial equations by radicals corresponds to the solvability of the associated Galois group. We proved the insolvability of the quintic, explained why certain constructions are impossible, and glimpsed the deep connection between symmetry and solvability.

Key achievements: Fundamental theorem of Galois theory, insolvability of the quintic, cyclotomic fields, proof of the fundamental theorem of algebra.

Chapter 25 — Modern Applications & Conclusion — Page 7

Volume V: Modern Abstractions

We ascended to the highest levels of abstraction. Modules generalized vector spaces over arbitrary rings. Category theory revealed universal properties and patterns across all of mathematics. Representation theory showed how abstract groups manifest as matrices. Algebraic number theory extended arithmetic beyond the integers, resolving Fermat's Last Theorem. And applications demonstrated the unreasonable effectiveness of these abstractions in the real world.

Key achievements: Module theory, exact sequences, category theory, functors and natural transformations, representation theory and characters, algebraic integers and class field theory, modern applications.

The Unity of Abstract Algebra

What have we really learned through this journey? Not merely a catalog of theorems and techniques, but a *way of thinking*—a mathematical worldview centered on three core principles:

1. Structure is Primary. Mathematics is not about objects but about relationships and structure. A group is not its elements but the patterns they form under operation. An isomorphism preserves structure while changing labels. We learned to see past superficial differences to underlying unity.

2. Abstraction Reveals Generality. By abstracting from specific examples to general principles, we gain power. The same theorem applies to seemingly different contexts because they share abstract structure. Abstraction isn't obscurity—it's clarity at a higher level.

3. Symmetry is Fundamental. From group actions to Galois groups to representation theory, symmetry emerges as the organizing principle. Noether's theorem in physics, normal extensions in Galois theory, invariant subspaces in representation theory—all express the same deep truth: symmetry dictates structure.

Connections to Other Mathematics

Abstract algebra does not stand alone. It interweaves with every branch of mathematics:

- **Topology:** Fundamental groups, homology groups, covering spaces—all algebraic invariants of topological spaces
- **Analysis:** Fourier analysis is representation theory of $\mathbb{R}$; harmonic analysis studies representations of locally compact groups
- **Geometry:** Transformation groups define geometries (Klein's Erlangen program); Lie groups are manifolds with group structure
- **Number Theory:** Galois groups, class field theory, modular forms—algebra and number theory are inseparable
- **Logic:** Model theory studies structures satisfying axioms; universal algebra abstracts common patterns

Modern mathematics is unified by algebraic thinking. Category theory provides the common language, and structure-preserving maps (morphisms) are the universal concept.

Future Directions: The Frontier of Research

Abstract algebra is not a closed subject but a living, growing field. Major open problems and active research areas include:

The Langlands Program. Launched by Robert Langlands in the 1960s, this is a vast network of conjectures connecting number theory, representation theory, and geometry. It generalizes class field theory and has been called "a grand unified theory of mathematics." Recent progress includes the proof of the fundamental lemma by Ngô Bảo Châu (Fields Medal 2010).

The Inverse Galois Problem. Does every finite group occur as the Galois group of some extension of $\mathbb{Q}$? Known to be true for many groups (symmetric groups, alternating groups, many others) but open in general. Connects group theory, field theory, and number theory.

Classification of Finite Simple Groups. The classification was completed (announced in 1983, completed ~2004) but the proof spans tens of thousands of pages across

hundreds of papers. Current work involves simplifying and verifying the proof, and understanding the sporadic groups more deeply. The Monster group and moonshine continue to reveal surprising connections.

Homological Algebra and Derived Categories. Extension of module theory to derived functors, spectral sequences, triangulated categories. Essential tools in algebraic geometry, representation theory, and mathematical physics.

Non-commutative Geometry. Developed by Alain Connes, this extends geometric concepts to non-commutative algebras, with applications to quantum physics and number theory.

Higher Category Theory. Categories of categories, ∞-categories, homotopy type theory—attempting to provide even more abstract foundations for mathematics.

The Unreasonable Effectiveness of Mathematics

Eugene Wigner asked in 1960: Why is mathematics so unreasonably effective in describing the physical world? Abstract algebra—born from pure thought about solving equations—now governs particle physics, encrypts communications, and enables space exploration.

Perhaps the answer lies in the nature of structure itself. Mathematics studies patterns and structure in their purest form. The physical universe exhibits structure—symmetries, conservation laws, quantization. Is it any wonder that the language of pure structure applies to physical reality?

Or perhaps, as the Platonists believe, mathematics describes an eternal realm of forms, and physical reality is merely its shadow. Either way, the deep applicability of abstract mathematics remains one of the great mysteries and miracles of existence.

> ⚡ **Greek Meditation** ⚡
>
> **The Philosopher's Ascent: From Cave to Sun**
>
> Plato's allegory: prisoners in a cave see only shadows,
> projections of reality on the wall.
> The philosopher breaks free, ascends to sunlight,
> sees forms themselves—the eternal truths.
>
> So too our journey:
> We began with concrete examples—shadows on the wall.
> Through abstraction we ascended,

> glimpsing the forms: groups, fields, categories.
> Now we see that physical reality itself
> is but a shadow of mathematical form.
>
> Symmetry, structure, transformation—
> these are not human inventions but eternal truths,
> waiting to be discovered, not created.

Final Reflections: The Beauty of Pure Thought

We close with thoughts on the nature of mathematics itself—what we have been doing these five volumes, and why it matters.

Mathematics as Discovery vs Invention. Do we discover mathematical truths that exist independently, or do we invent consistent systems of axioms and derive consequences? The Platonist sees mathematics as discovery—groups and fields exist in a realm beyond space and time. The formalist sees invention—we choose axioms and play out their implications.

Perhaps both are true. We invent the language, the notation, the axioms. But once chosen, the theorems follow necessarily. We discover consequences we never anticipated, connections we never planned. The formalist invents chess; the Platonist discovers that white has an advantage.

Beauty, Truth, and Understanding. Mathematicians speak of beautiful proofs, elegant theorems. This is not metaphor—mathematical beauty is as real as musical or visual beauty. When we prove Galois theory's fundamental theorem, connecting group structure to field structure, we experience genuine aesthetic pleasure.

What makes mathematics beautiful? Simplicity from complexity. Unity from diversity. The unexpected connection. The moment when disparate pieces suddenly align. These are not accidents but glimpses of the deep order underlying reality.

Why Study Abstract Algebra? Not everyone will use Sylow theorems or Galois groups in daily life. But the habits of thought cultivated by abstract algebra—precise reasoning, creative problem-solving, pattern recognition, the ability to work with abstraction—these are universally valuable.

More than that: to study mathematics is to participate in one of humanity's greatest collective achievements. The theorems in these five volumes represent millennia of

human thought. From Babylon to Greece to the Islamic Golden Age to Renaissance Europe to the modern era—these are treasures of civilization.

A Gratitude

To the pioneers who forged these paths:
Évariste Galois, dying at 20, revealing the connection between symmetry and solvability.
Niels Henrik Abel, proving the impossibility of the quintic.
Richard Dedekind, creating ideal theory to save unique factorization.
Emmy Noether, whose abstraction revolutionized algebra and physics.
And countless others whose insights illuminate our journey.

We are dwarves standing on the shoulders of giants.

The Journey Continues

This is not an ending but a beginning. The tools acquired here open doors to advanced mathematics: algebraic geometry, algebraic topology, algebraic number theory, homological algebra, representation theory of Lie groups, and more.

Each reader will take away something different. Some will pursue research mathematics. Others will apply these ideas in science, engineering, or technology. Still others will simply carry forward an appreciation for the beauty of pure thought.

Whatever path you choose, you have touched something eternal. The groups, rings, and fields we studied will continue existing long after we are gone. The theorems we proved were true before humans walked the Earth and will remain true after the stars grow cold.

In studying mathematics, we commune with something larger than ourselves—the structure of reality itself.

Practice Problems

Problem 25.1. With $n = 35, e = 5$, find the private key d and decrypt the message $c = 10$.

> ***Solution.*** We have $n = 35 = 5 \cdot 7$, so $\phi(n) = 4 \cdot 6 = 24$. Find $d = e^{-1} \pmod{24}$:
>
> $$5d \equiv 1 \pmod{24}.$$
>
> Trying: $5 \cdot 5 = 25 \equiv 1 \pmod{24}$. So $d = 5$.

Decrypt: $m \equiv 10^5 \pmod{35}$. Compute: $10^2 = 100 \equiv 30 \pmod{35}$, $10^4 \equiv 30^2 = 900 \equiv 25 \pmod{35}$, $10^5 \equiv 25 \cdot 10 = 250 \equiv 5 \pmod{35}$.

Thus $m = 5$.

Problem 25.2. Why is it essential that $\gcd(e, \phi(n)) = 1$ in RSA?

Solution. If $\gcd(e, \phi(n)) \neq 1$, then e has no multiplicative inverse modulo $\phi(n)$, so we cannot compute $d = e^{-1}$. Without d, decryption is impossible.

Moreover, if $\gcd(e, \phi(n)) = g > 1$, then encryption $c = m^e$ might not be injective—different messages could encrypt to the same ciphertext, making decryption ambiguous.

Problem 25.3. Explain how Euler's theorem ($a^{\phi(n)} \equiv 1 \pmod{n}$ for $\gcd(a, n) = 1$) follows from Lagrange's theorem.

Solution. The group $(\mathbb{Z}/n\mathbb{Z})^*$ consists of residue classes coprime to n, with multiplication mod n. This group has order $\phi(n)$.

By Lagrange's theorem, for any element $a \in (\mathbb{Z}/n\mathbb{Z})^*$, the order of a divides $|G| = \phi(n)$. Thus $a^{\phi(n)} = e$, i.e., $a^{\phi(n)} \equiv 1 \pmod{n}$.

Problem 25.4. Show that the set of points on an elliptic curve $y^2 = x^3 + ax + b$ over $\mathbb{R}$ forms a group.

Solution. (Sketch) The group operation: given points P, Q, draw a line through them, find the third intersection point R with the curve, and reflect across the x-axis to get $P + Q$.

Identity: point at infinity $\mathcal{O}$. Inverse: $-P = (x, -y)$ for $P = (x, y)$. Associativity requires proof using Bézout's theorem from algebraic geometry—any two cubics intersect in 9 points (counting multiplicity).

Problem 25.5. Encode the Reed-Solomon code with $k = 3, n = 5$ over $\mathbb{F}_7$, encode the message $(2, 3, 1)$.

Solution. The message $(2, 3, 1)$ corresponds to polynomial $f(x) = 2 + 3x + x^2$. Evaluate at 5 points, say $\alpha_i = i$ for $i = 0, 1, 2, 3, 4$:

$$\begin{aligned} f(0) &= 2 \\ f(1) &= 2 + 3 + 1 = 6 \\ f(2) &= 2 + 6 + 4 = 12 \equiv 5 \pmod{7} \\ f(3) &= 2 + 9 + 9 = 20 \equiv 6 \pmod{7} \\ f(4) &= 2 + 12 + 16 = 30 \equiv 2 \pmod{7} \end{aligned}$$

Codeword: $(2, 6, 5, 6, 2)$.

Problem 25.6. What is the minimum distance of an $[n, k]$ Reed-Solomon code?

> ***Solution.*** The minimum distance is $d = n - k + 1$.
>
> **Reason:** Two distinct codewords correspond to polynomials $f(x) \neq g(x)$ of degree $< k$. Their difference $h(x) = f(x) - g(x)$ has degree $< k$, so at most $k - 1$ roots. Thus f and g agree at most $k - 1$ positions, differing in at least $n - (k - 1) = n - k + 1$ positions.

Problem 25.7. Explain why cyclic codes correspond to ideals in $\mathbb{F}_q[x]/(x^n - 1)$.

> ***Solution.*** Identify codewords $(c_0, c_1, \ldots, c_{n-1}) \in \mathbb{F}_q^n$ with polynomials $c(x) = c_0 + c_1 x + \cdots + c_{n-1} x^{n-1}$ in $\mathbb{F}_q[x]/(x^n - 1)$.
>
> A cyclic shift corresponds to multiplication by x: $x \cdot c(x) = c_{n-1} + c_0 x + \cdots + c_{n-2} x^{n-1}$ (using $x^n \equiv 1$).
>
> A cyclic code C is closed under cyclic shifts, hence under multiplication by x, making C an ideal in the quotient ring.

Problem 25.8. State Noether's theorem and give an example of symmetry → conservation law.

> ***Solution.*** **Noether's theorem:** Every continuous symmetry of a physical system corresponds to a conserved quantity.
>
> **Example:** Rotational symmetry (system unchanged by rotation) implies angular momentum is conserved. If physics is the same in all directions, angular momentum $\mathbf{L}$ cannot change.

Problem 25.9. Why are there exactly 230 space groups in 3 dimensions?

> ***Solution.*** Space groups combine point groups (rotational symmetries, 32 types) with translations (lattices, 14 Bravais lattices). The combination must satisfy compatibility conditions.
>
> Classification involves:
>
> - 7 crystal systems (triclinic, monoclinic, etc.)
> - 32 crystallographic point groups
> - 14 Bravais lattices
> - Adding screw axes and glide planes
>
> Exhaustive enumeration (done by Fedorov and Schoenflies independently) gives exactly 230 distinct space groups.

Problem 25.10. What is the gauge group of the Standard Model and what does each factor represent?

> ***Solution.*** $SU(3) \times SU(2) \times U(1)$:
>
> - $SU(3)$: color symmetry of strong force (quantum chromodynamics). Quarks in fundamental rep, gluons in adjoint (8D)
> - $SU(2)$: weak isospin. Left-handed fermions form doublets
> - $U(1)$: weak hypercharge. After electroweak symmetry breaking, combines with $SU(2)$ to give electromagnetism

Problem 25.11. Why does Shor's algorithm threaten RSA?

> ***Solution.*** Shor's algorithm factors integers in polynomial time on a quantum computer (classical algorithms require exponential time). Since RSA security relies on the hardness of factoring $n = pq$, a quantum computer running Shor's algorithm breaks RSA completely.
>
> The algorithm uses quantum Fourier transform—essentially representation theory of cyclic groups—to find the period of $a^x \bmod n$, which leads to factorization.

Problem 25.12. Explain the connection between representation theory and character theory.

> ***Solution.*** A representation $\rho : G \to GL(V)$ assigns matrices to group elements. The character $\chi(g) = \mathrm{Tr}(\rho(g))$ captures essential information.
>
> Characters are much simpler than representations (functions vs matrices), yet characters of irreducible representations form an orthonormal basis for class functions. Remarkably, two representations are equivalent iff they have the same character—the character determines the representation up to isomorphism!

Problem 25.13. What is the significance of the Langlands program?

> ***Solution.*** The Langlands program is a vast web of conjectures connecting:
>
> - Galois representations (number theory)
> - Automorphic forms (representation theory)
> - L-functions (analysis)
>
> It vastly generalizes class field theory from abelian to non-abelian extensions. Progress includes proving Fermat's Last Theorem (via

modularity theorem, a special case of Langlands).

Problem 25.14. State the inverse Galois problem.

> ***Solution.*** **Question:** Does every finite group G occur as $\mathrm{Gal}(K/\mathbb{Q})$ for some Galois extension $K/\mathbb{Q}$?
>
> **Known:** True for S_n, A_n, many solvable groups, many sporadic simple groups.
>
> **Open:** General case remains unresolved. It's one of the major unsolved problems in algebra.

Problem 25.15. Summarize the classification of finite simple groups.

> ***Solution.*** Every finite simple group falls into one of four categories:
>
> 1. **Cyclic groups** C_p of prime order
> 2. **Alternating groups** A_n for $n \geq 5$
> 3. **Groups of Lie type:** analogues of Lie groups over finite fields (e.g., $PSL_n(\mathbb{F}_q)$)
> 4. **26 sporadic groups:** exceptional groups including the Monster M (order $\sim 8 \times 10^{53}$)
>
> The proof spans ~15,000 pages. Simplification efforts continue.

Problem 25.16. How does representation theory apply to quantum mechanics?

> ***Solution.*** Quantum states transform under symmetry groups (rotations, Lorentz transformations, etc.). Irreducible representations correspond to physically distinct particle types or states.
>
> Examples:
>
> - Spin-½ particles: irreducible rep of SU(2)
> - Angular momentum: reps of SO(3)
> - Selection rules: determined by Clebsch-Gordan coefficients

Problem 25.17. What makes the Monster group special?

> ***Solution.*** The Monster M is the largest sporadic simple group, with order:
>
> $$|M| = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71.$$

> Mysterious connections to physics (string theory) and number theory (modular functions) via "monstrous moonshine" (proven by Borcherds, Fields Medal 1998). The Monster seems to hint at deep structures we don't yet understand.

Problem 25.18. Explain the connection between QR codes and Reed-Solomon codes.

> ***Solution.*** QR codes use Reed-Solomon error correction to remain scannable even when partially damaged. The redundancy allows reconstruction of the data even if up to 30% of the code is obscured.
>
> Different QR code error correction levels (L, M, Q, H) correspond to different RS code parameters, trading data capacity for error resilience.

Problem 25.19. Why is category theory called "abstract nonsense"?

> ***Solution.*** The term (used affectionately!) reflects that category theory operates at a very high level of abstraction—it studies patterns common to all mathematical structures, not specific objects.
>
> Yet this "nonsense" is incredibly powerful: universal properties, functoriality, and natural transformations clarify and unify vast areas of mathematics. What seems abstract becomes indispensable for organizing modern mathematics.

Problem 25.20. (Synthesis) Trace how one concept from Volume I connects through to Volume V.

> ***Solution.*** Example: **Group homomorphisms**
>
> - **Volume I:** Defined structure-preserving maps $\phi : G \to H$
> - **Volume II:** Extended to ring homomorphisms
> - **Volume III:** Field homomorphisms (embeddings) crucial for extensions
> - **Volume IV:** Galois group elements are field automorphisms (special homomorphisms)
> - **Volume V:** Module homomorphisms, functors (morphisms of categories), representations (homomorphisms $G \to GL(V)$)
>
> The core idea—preserving structure—remains constant while generalizing across all contexts!

EPILOGUE

"Mathematics, rightly viewed, possesses not only truth, but supreme beauty—
a beauty cold and austere, like that of sculpture,
without appeal to any part of our weaker nature,
without the gorgeous trappings of painting or music,
yet sublimely pure, and capable of a stern perfection
such as only the greatest art can show."
— Bertrand Russell

Dear Reader,

If you have journeyed with us through all five volumes, you have accomplished something remarkable. You have scaled one of the great intellectual peaks of human civilization. The mathematics contained in these pages represents centuries of collective human effort—insights purchased through lifetimes of dedication by some of history's greatest minds.

Abstract algebra is not easy. It demands rigorous thinking, careful reasoning, and patience with abstraction. There were likely moments of frustration, of confusion, of wanting to give up. Yet you persevered. And in doing so, you have transformed your mind—you now see patterns invisible to most, understand structures beyond ordinary intuition, and speak a language shared by mathematicians across all cultures and eras.

You have learned more than theorems and proofs. You have learned a way of thinking—a methodology that applies far beyond mathematics. The discipline of precise definition, the art of proof, the power of abstraction, the insight of viewing objects through their morphisms—these are gifts that will serve you in any intellectual endeavor.

You have also connected yourself to a tradition stretching back millennia. When you work with groups, you walk in the footsteps of Galois, tragic genius dead at twenty. When you study ideals, you follow Dedekind and Noether, pioneers who dared to think more abstractly than their contemporaries. When you explore Galois theory, you witness one of mathematics' greatest triumphs—the synthesis that finally answered questions posed in antiquity.

The journey does not end here. These five volumes are not a conclusion but a foundation. If you wish to pursue mathematics further, you are now equipped to study algebraic geometry, algebraic topology, algebraic number theory, representation theory of Lie groups, homological algebra, and countless other fields. Each opens new vistas, new challenges, new beauty.

Even if you never study mathematics again, you have gained something precious—an appreciation for the extraordinary edifice that human minds have constructed through pure thought. Mathematics is one of our species' greatest achievements, a cathedral built of ideas rather than stone, existing not in physical space but in the realm of mind.

And unlike physical monuments, mathematical truths are eternal. The theorems you have learned will remain true forever. Long after our civilization has passed, long after the Earth itself has been consumed by the dying sun, the truth that groups have homomorphic images, that polynomials factor uniquely in UFDs, that Galois groups determine solvability—these will remain, written not in stone but in the structure of reality itself.

Chapter 25 — Modern Applications & Conclusion — Page 13

In studying abstract algebra, you have done something profoundly human and profoundly noble. You have sought to understand, not for practical gain but for the sake of understanding itself. This is what separates us from other animals—not tool use, not language, but the pure love of knowledge, the yearning to comprehend the patterns underlying existence.

Go forward with confidence. Whether you become a research mathematician, apply these ideas in science or engineering, teach others, or simply carry forward an appreciation for mathematical beauty—you have accomplished something worthwhile. You have exercised your mind at its highest capacity, engaging with ideas that demand the best of human intellect.

> **☯ Final Zen Meditation ☯**
>
> **The Journey's End: Returning Home, Having Never Left**
>
> Before we began, mountains were mountains.
> During the journey, mountains were no longer mountains—
> they became structures, patterns, abstract categories.
> Now, at the end, mountains are mountains again—
> but we are not the same.
>
> We left to travel far, seeking knowledge.
> We return home, having never left,
> for all mathematics was always within us,
> waiting to be recognized.
>
> The circle completes itself.
> The river returns to the ocean.
> The student becomes the teacher.
> And the eternal dance of form and structure

> continues, as it always has,
> as it always will.

May your journey continue,
may your mind remain curious,
and may you find beauty in structure,
truth in abstraction,
and wonder in the eternal patterns of mathematics.

τέλος

(The End, and The Beginning)

"Symmetry is one idea by which man through the ages
has tried to comprehend and create order, beauty, and perfection."

— Hermann Weyl

☥ End of Volume V ☥
End of the Five-Volume Series

Volume 6: What Was Always Visible

Ἀποκάλυψις

Revelation is not discovery of the hidden,
but recognition of the present—
learning to see what was always before us,
waiting for eyes trained to perceive.

A Zen student travels far seeking enlightenment, only to return home exhausted and discover that his ordinary mind, his daily life, his simple awareness—this was what he sought all along. The extraordinary is not elsewhere. It is here, in the ordinary, once we learn to see.

This final volume returns to familiar ground with transformed vision. The vector spaces of undergraduate linear algebra now reveal depths we had not noticed: dual spaces that shadow every vector space, functionals that map vectors to scalars, a whole geometry of linear transformations we overlooked while focusing on matrix mechanics. Nothing has been added. We simply learned to see what was present from the beginning.

On Dual Spaces:

Given a vector space V, its dual V^* consists of all linear functionals —linear maps from V to the scalar field. This seems technical, abstract. But here's the insight: every vector space contains within it a perfect shadow self. Every geometric object has a functional twin.

Position and momentum in quantum mechanics are dual to each other. Columns and rows of matrices are dual. The wave and the particle, in some deep sense, are dual aspects of a single reality.

The double dual V^{**} is naturally isomorphic to V for finite-dimensional spaces—a space reflected in its shadow, reflected again, returns to itself. This is not mathematical coincidence. It's structure revealing its own self-consistency, form acknowledging itself in its functional image.

On Lie Groups and Continuous Symmetry:

We studied discrete groups—permutations, symmetries of polygons, finite algebraic structures. But the universe is continuous. Rotations don't happen in discrete jumps; they flow smoothly. Lie groups are group structure merged with continuous geometry: groups that are also smooth manifolds, where multiplication and inversion are continuous operations.

The rotation group SO(3) governs rotational symmetry in three-dimensional space. The unitary group U(n) governs quantum mechanics. The Lorentz group governs special relativity. These are not applications grafted onto algebra—they are algebra revealed in its natural habitat, structure manifesting in physical law.

Here's what startles: the Lie group's structure is encoded in its Lie algebra, the tangent space at the identity element. Infinite-dimensional continuous symmetry compresses into finite-dimensional infinitesimal structure. We can study smooth groups by studying flat vector spaces with a bracket operation. The global is contained in the local; the curved is remembered in the flat.

On Category Theory Applied:

Category theory, so abstract in Volume 5, here shows its power. Universal properties—initial objects, terminal objects, products, coproducts—turn out to be how the universe constructs things. Direct sums, tensor products, quotient structures, free objects—all are categorical constructions, specific instances of universal patterns.

The Yoneda lemma asserts that an object is completely determined by the morphisms into it (or out from it). This seems technical, but encoded within is a profound insight: a thing is known by its relationships, defined by how it interacts with all else. There is no isolated essence, no substance independent of relation. Being is relational through and through.

On Physics and Algebra:

The symmetries of quantum mechanics are group representations on Hilbert spaces. Conservation laws—energy, momentum, angular momentum—arise from symmetries through Noether's theorem: each continuous symmetry of a physical system corresponds to a conserved quantity. This is not physics using mathematics as a tool. This is mathematics revealing that physical law is fundamentally algebraic structure.

Gauge theories, which describe electromagnetic, weak, and strong forces, are fiber bundles with Lie group structure. The geometry of spacetime in general relativity is Riemannian geometry, fundamentally built on the calculus of differential forms and tensors —algebraic objects. String theory requires algebraic geometry, category theory, representation theory.

The universe, it seems, is not merely describable by mathematics—it may *be* mathematical structure, algebra incarnate, category theory

made flesh and field.

On Returning Home:

Zen adept Qingyuan's famous verse traces understanding's arc: "Before I studied Zen, mountains were mountains and waters were waters. When I first studied Zen, mountains were no longer mountains and waters were no longer waters. But after I understood Zen, mountains were again mountains and waters were again waters."

We began with concrete mathematics—numbers, equations, operations. Through abstraction, these dissolved into formal structures, axioms, categories. Now we return: numbers are again numbers, but we see them as instances of fields, rings, modules. Symmetry is again visible symmetry, but we recognize it as group action. The ordinary has become transparent to the extraordinary. The familiar reveals depths we couldn't have imagined at the start.

This is mathematical maturity: not transcending the concrete for the abstract, but seeing the abstract *through* the concrete, perceiving universal structure in particular instance, recognizing that the ordinary and the profound are not separate realms but different modes of seeing the same reality.

> **Greek Wisdom — Ἑλληνική Σοφία**
>
> *"Τὰ ὄντα ἰέναι τε πάντα καὶ μένειν οὐδέν" — All things flow, yet forms remain*
>
> Heraclitus taught that change is fundamental, yet patterns persist. This volume reveals pattern recurring across domains:

algebra manifesting in physics, category theory unifying disparate fields, abstract structure finding concrete expression. These connections were always present, inherent in reality's structure, waiting for mathematics to develop eyes capable of seeing them. Mathematical truth is not invented but uncovered, not constructed but revealed. The forms were always there.

Zen Reflection — 禅の省察

Ordinary mind is the way. Before enlightenment, chop wood and carry water. After enlightenment, chop wood and carry water.

After all the abstraction, all the theory, all the generalization, we return to find the familiar transformed—not changed externally, but seen clearly for the first time. The vector spaces we studied as undergraduates now reveal dual spaces and tensor products we hadn't noticed. The symmetries visible in everyday objects are Lie groups manifesting continuously. The physics we learned as formulas are categorical structures in disguise.

Nothing has changed, yet everything is new. This is *悟り* (satori)—awakening to what was always true. The profound was never hidden, only unrecognized. Mathematical maturity is learning to see what was always visible, to recognize the universal in the particular, the abstract in the concrete. The way was always here, in ordinary mathematics, in daily structure, in the simple act of perceiving pattern.

This volume invites return, not departure. Here are connections between algebra and the wider mathematical cosmos, applications to physics where abstract structure reveals itself as natural law, depths beneath surfaces we thought we understood. What was always visible awaits your recognition—patient as light, present as awareness, eternal as structure itself.

Part A: Dual Spaces and Linear Functionals

PART A

✦

DUAL SPACES & DUALITY

The concept of duality is one of the most profound and elegant ideas in mathematics. For every algebraic structure, there often exists a "mirror" structure—its dual—that reflects and illuminates hidden properties of the original. This principle appears throughout mathematics: dual vector spaces in linear algebra, adjoint operators in functional analysis, Pontryagin duality in harmonic analysis, and categorical duality in abstract settings.

In this part, we explore various manifestations of duality, beginning with the most fundamental case: dual vector spaces. We'll discover that duality is not merely a mathematical curiosity but a powerful tool that reveals deep symmetries and provides new perspectives on familiar concepts.

★ ✦ ★

A.1 Dual Vector Spaces

Let V be a vector space over a field F. The most natural way to probe the structure of V is through *linear functionals*—linear maps from V to the field F itself.

> **Definition**
>
> Definition A.1.1 (Dual Space).
> Let V be a vector space over a field F. The dual space of V, denoted V^*, is defined as:
>
> $$V^* = \mathrm{Hom}_F(V, F) = \{\varphi : V \to F \mid \varphi \text{ is linear}\}$$
>
> Elements of V^* are called linear functionals or covectors.

The dual space V^* is itself a vector space over F with operations defined pointwise:

- Addition: $(\varphi + \psi)(v) = \varphi(v) + \psi(v)$ for all $v \in V$
- Scalar multiplication: $(c\varphi)(v) = c \cdot \varphi(v)$ for all $c \in F, v \in V$

Example

Example A.1.2 (Dual of $\mathbb{R}^n$).
Consider $V = \mathbb{R}^n$ with the standard basis $\{e_1, e_2, \ldots, e_n\}$. Every linear functional $\varphi : \mathbb{R}^n \to \mathbb{R}$ can be represented as:

$$\varphi(x_1, x_2, \ldots, x_n) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

for some fixed scalars $a_1, a_2, \ldots, a_n \in \mathbb{R}$. This shows that $(\mathbb{R}^n)^* \cong \mathbb{R}^n$ as vector spaces. However, this isomorphism is *not canonical*—it depends on the choice of basis.

Example

Example A.1.3 (Coordinate Functionals).
Let V be a vector space with basis $\{e_1, e_2, \ldots, e_n\}$. For each index i, define the i-th coordinate functional $e_i^* : V \to F$ by:

$$e_i^*(c_1 e_1 + c_2 e_2 + \cdots + c_n e_n) = c_i$$

These functionals extract the i-th coordinate of a vector with respect to the given basis.

Theorem

Theorem A.1.4 (Dual Basis).
Let V be a finite-dimensional vector space over F with basis $\mathcal{B} = \{e_1, e_2, \ldots, e_n\}$. Then the set $\mathcal{B}^* = \{e_1^*, e_2^*, \ldots, e_n^*\}$ of coordinate functionals forms a basis for V^*, called the dual basis. Moreover, these functionals satisfy:

$$e_i^*(e_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Proof. Proof. We must show that $\mathcal{B}^*$ is linearly independent and spans V^*.

Linear Independence: Suppose $\sum_{i=1}^{n} c_i e_i^* = 0$ (the zero functional). Evaluating at e_j:

$$0 = \left(\sum_{i=1}^{n} c_i e_i^*\right)(e_j) = \sum_{i=1}^{n} c_i e_i^*(e_j) = \sum_{i=1}^{n} c_i \delta_{ij} = c_j$$

Thus all $c_j = 0$, proving linear independence.

Spanning: Let $\varphi \in V^*$ be arbitrary. For any $v = \sum_{j=1}^{n} x_j e_j \in V$:

$$\varphi(v) = \varphi\left(\sum_{j=1}^{n} x_j e_j\right) = \sum_{j=1}^{n} x_j \varphi(e_j)$$

Let $a_i = \varphi(e_i)$. Then:

$$\varphi(v) = \sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n} a_i e_i^*(v) = \left(\sum_{i=1}^{n} a_i e_i^*\right)(v)$$

Therefore $\varphi = \sum_{i=1}^{n} a_i e_i^*$, proving that $\mathcal{B}^*$ spans V^*. ∎ ∎

Corollary A.1.5 (Dimension of Dual Space).
If V is a finite-dimensional vector space over F, then:

$$\dim(V^*) = \dim(V)$$

Example

Example A.1.6 (Integration as a Functional).
Let $V = C[0, 1]$ be the vector space of continuous real-valued functions on $[0, 1]$. Define $\varphi : V \to \mathbb{R}$ by:

$$\varphi(f) = \int_0^1 f(x)\, dx$$

This is a linear functional: $\varphi(f + g) = \varphi(f) + \varphi(g)$ and $\varphi(cf) = c\varphi(f)$. Thus integration defines an element of the dual space $(C[0, 1])^*$.

Example

Example A.1.7 (Point Evaluation).
Let V be a space of functions and fix a point x_0 in the domain. Define $\delta_{x_0} : V \to \mathbb{R}$ by:

$$\delta_{x_0}(f) = f(x_0)$$

This "Dirak delta" functional evaluates a function at a specific point. It is linear:
$\delta_{x_0}(f + g) = f(x_0) + g(x_0)$ and $\delta_{x_0}(cf) = cf(x_0)$.

Non-Canonical Nature of Isomorphisms

While $V \cong V^*$ for finite-dimensional spaces (both have the same dimension), this isomorphism is *not canonical*—it depends on choosing a basis. Different bases yield different isomorphisms. This subtlety becomes crucial in advanced applications.

✦

A.2 Double Dual and Reflexivity

If we can take the dual of V to get V^*, what happens if we take the dual again? The double dual $V^{**} = (V^*)^*$ reveals a remarkable property.

> **Definition**
>
> Definition A.2.1 (Double Dual).
> The double dual (or bidual) of a vector space V is:
>
> $$V^{**} = (V^*)^* = \mathrm{Hom}_F(V^*, F)$$

There is a natural way to view elements of V as elements of V^{**}:

> **Theorem**
>
> Theorem A.2.2 (Natural Embedding into Double Dual).
> There exists a natural (canonical) linear map $\iota : V \to V^{**}$ defined by:
>
> $$\iota(v)(\varphi) = \varphi(v)$$
>
> for all $v \in V$ and $\varphi \in V^*$. This map is always injective.

Proof. Proof. First, we verify that $\iota(v)$ is indeed a linear functional on V^*. For $\varphi, \psi \in V^*$ and $c \in F$:

$$\iota(v)(\varphi + \psi) = (\varphi + \psi)(v) = \varphi(v) + \psi(v) = \iota(v)(\varphi) + \iota(v)(\psi)$$
$$\iota(v)(c\varphi) = (c\varphi)(v) = c\varphi(v) = c \cdot \iota(v)(\varphi)$$

Thus $\iota(v) \in V^{**}$.

Next, we show ι is linear. For $v, w \in V$ and $c \in F$:

$$\iota(v + w)(\varphi) = \varphi(v + w) = \varphi(v) + \varphi(w) = \iota(v)(\varphi) + \iota(w)(\varphi)$$
$$\iota(cv)(\varphi) = \varphi(cv) = c\varphi(v) = c \cdot \iota(v)(\varphi)$$

Finally, injectivity: suppose $\iota(v) = 0$ (the zero functional). Then $\iota(v)(\varphi) = 0$ for all $\varphi \in V^*$. In particular, $\varphi(v) = 0$ for all $\varphi \in V^*$. If $v \neq 0$, we can extend $\{v\}$ to a basis and construct a functional φ with $\varphi(v) = 1$, contradicting $\varphi(v) = 0$. Thus $v = 0$. ∎ ∎

Theorem

Theorem A.2.3 (Reflexivity of Finite-Dimensional Spaces).
If V is finite-dimensional, then $\iota : V \to V^{**}$ is an isomorphism. We say V is reflexive.

Proof. Proof. Since $\dim(V^*) = \dim(V)$ and $\dim(V^{**}) = \dim(V^*)$, we have $\dim(V^{**}) = \dim(V)$. An injective linear map between finite-dimensional spaces of the same dimension is automatically surjective, hence an isomorphism. ∎ ∎

Example

Example A.2.4 (Non-Reflexive Infinite-Dimensional Space).
Let $V = c_{00}$ be the space of sequences $(a_1, a_2, a_3, \ldots)$ with only finitely many nonzero terms. Then $V^* \cong \ell^1$ (the space of absolutely summable sequences), but $V^{**} \cong \ell^\infty$ (the space of bounded sequences). Since $c_{00} \subsetneq \ell^\infty$, we have $V \subsetneq V^{**}$ (the inclusion is proper), so V is not reflexive.

The key distinction: the natural embedding $\iota : V \to V^{**}$ is always injective, but only surjective (hence an isomorphism) for finite-dimensional spaces or certain "nice" infinite-dimensional spaces (like Hilbert spaces).

✦

A.3 Transpose and Adjoint

Linear transformations between vector spaces induce natural transformations between their duals, but in the *opposite direction*. This is our first taste of contravariance.

Definition

Definition A.3.1 (Transpose/Dual Map).
Let $T : V \to W$ be a linear transformation. The transpose (or dual map) $T^* : W^* \to V^*$ is defined by:

$$(T^*\varphi)(v) = \varphi(T(v))$$

for all $\varphi \in W^*$ and $v \in V$.

Notice the direction reversal: T goes from V to W, but T^* goes from W^* to V^*. We can visualize this as:

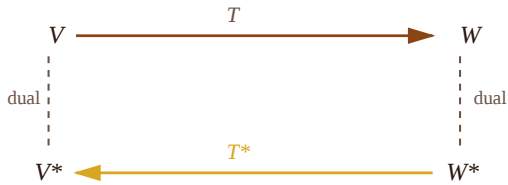

Figure A.1: The transpose map T^* goes in the opposite direction from T.

Theorem

Theorem A.3.2 (Properties of Transpose).
Let $S : U \to V$ and $T : V \to W$ be linear transformations. Then:

1. $(T \circ S)^* = S^* \circ T^*$ (order reverses)
2. $(\mathrm{id}_V)^* = \mathrm{id}_{V^*}$
3. If T is invertible, then $(T^{-1})^* = (T^*)^{-1}$
4. $(T^*)^* = \iota_W \circ T \circ \iota_V^{-1}$ (for finite-dimensional V, W, essentially T)

Proof. Proof of (1). For any $\varphi \in W^*$ and $u \in U$:

$$((T \circ S)^*\varphi)(u) = \varphi((T \circ S)(u)) = \varphi(T(S(u)))$$
$$= (T^*\varphi)(S(u)) = (S^*(T^*\varphi))(u) = ((S^* \circ T^*)\varphi)(u)$$

Thus $(T \circ S)^* = S^* \circ T^*$. ∎ ∎

Matrix Representation

In the finite-dimensional case, if $T : V \to W$ has matrix representation $[T]$ with respect to chosen bases, then $T^* : W^* \to V^*$ has matrix representation $[T]^T$ (the transpose matrix) with respect to the dual bases.

Example

Example A.3.3 (Transpose of a Matrix).
Consider $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by the matrix:

$$[T] = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$$

Then $T^* : (\mathbb{R}^3)^* \to (\mathbb{R}^2)^*$ has matrix:

$$[T^*] = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

✦

A.4 Annihilators

The dual space provides a natural way to study subspaces through their annihilators.

Definition

Definition A.4.1 (Annihilator).
Let $S \subseteq V$ be a subset. The annihilator of S is:

$$S^\perp = \{\varphi \in V^* : \varphi(v) = 0 \text{ for all } v \in S\}$$

Similarly, for $T \subseteq V^*$, we define:

$$T_\perp = \{v \in V : \varphi(v) = 0 \text{ for all } \varphi \in T\}$$

Theorem

Theorem A.4.2 (Properties of Annihilators).
Let V be finite-dimensional and $W \subseteq V$ a subspace. Then:

1. $W^\perp$ is a subspace of V^*
2. $\dim(W) + \dim(W^\perp) = \dim(V)$
3. $(W^\perp)_\perp = W$ (under the identification $V \cong V^{**}$)
4. If $W_1 \subseteq W_2$, then $W_2^\perp \subseteq W_1^\perp$

Proof. Proof of (2). Let $\{e_1, \dots, e_k\}$ be a basis for W, and extend it to a basis $\{e_1, \dots, e_k, e_{k+1}, \dots, e_n\}$ of V. Consider the dual basis $\{e_1^*, \dots, e_n^*\}$ of V^*.

Claim: $W^\perp = \operatorname{span}\{e_{k+1}^*, \dots, e_n^*\}$.

($\supseteq$) For $i > k$ and $j \leq k$: $e_i^*(e_j) = \delta_{ij} = 0$. Thus $e_i^* \in W^\perp$.

($\subseteq$) If $\varphi = \sum_{i=1}^n c_i e_i^* \in W^\perp$, then for each $j \leq k$:

$$0 = \varphi(e_j) = \sum_{i=1}^n c_i e_i^*(e_j) = c_j$$

Thus $c_1 = \cdots = c_k = 0$, so $\varphi \in \operatorname{span}\{e_{k+1}^*, \dots, e_n^*\}$.

Therefore $\dim(W^\perp) = n - k$, and $\dim(W) + \dim(W^\perp) = k + (n-k) = n = \dim(V)$. ∎

∎

Example

Example A.4.3 (Annihilator in $\mathbb{R}^3$).
Let $W = \operatorname{span}\{(1,0,0), (0,1,0)\}$ be the xy-plane in $\mathbb{R}^3$. Then:

$$W^\perp = \{\varphi \in (\mathbb{R}^3)^* : \varphi(x, y, 0) = 0 \text{ for all } x, y\}$$

This consists of functionals that "only see the z-coordinate": $\varphi(x,y,z) = cz$ for some $c \in \mathbb{R}$. Indeed, $\dim(W^\perp) = 1 = 3 - 2$.

Application to Linear Systems

Annihilators provide insight into solving systems of linear equations. The solution set of $T(v) = 0$ is precisely $(\operatorname{Im}(T^*))_\perp$.

✦

A.5 Dual Modules

The concept of duality extends beyond vector spaces to modules over rings.

Definition

Definition A.5.1 (Dual Module).
Let M be a (left) R-module. The dual module is:

$$M^* = \mathrm{Hom}_R(M, R)$$

This is naturally a right R-module: for $\varphi \in M^*$ and $r \in R$, define $(\varphi r)(m) = \varphi(m) \cdot r$.

Unlike vector spaces, dual modules can behave quite differently:

Example

Example A.5.2 (Dual of $\mathbb{Z}/n\mathbb{Z}$).
Consider $M = \mathbb{Z}/n\mathbb{Z}$ as a $\mathbb{Z}$-module. Then:

$$M^* = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0$$

This is because any homomorphism $\varphi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}$ must satisfy $n\varphi(\overline{1}) = \varphi(\overline{n}) = \varphi(\overline{0}) = 0$, forcing $\varphi(\overline{1}) = 0$ (since $\mathbb{Z}$ is torsion-free), hence $\varphi = 0$.

Theorem

Theorem A.5.3 (Dual of Free Module).
If M is a free R-module of finite rank n, then M^* is also free of rank n. Specifically, if $M \cong R^n$, then $M^* \cong R^n$ (as right R-modules).

The dual module construction connects to the theory of projective modules: a finitely generated module M is projective if and only if M^* is flat.

✦

A.6 Pontryagin Duality

One of the most beautiful generalizations of duality occurs in harmonic analysis: Pontryagin duality for locally compact abelian groups.

Definition

Definition A.6.1 (Character Group).
Let G be a locally compact abelian group. A character of G is a continuous group homomorphism $\chi : G \to \mathbb{T}$, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the circle group. The character group (or Pontryagin dual) is:

$$\widehat{G} = \{\chi : G \to \mathbb{T} : \chi \text{ continuous homomorphism}\}$$

with pointwise multiplication: $(\chi_1 \chi_2)(g) = \chi_1(g)\chi_2(g)$.

Theorem

Theorem A.6.2 (Pontryagin Duality Theorem).
Let G be a locally compact abelian group. Then:

1. $\widehat{G}$ is also a locally compact abelian group
2. The natural map $\iota : G \to \widehat{\widehat{G}}$ given by $\iota(g)(\chi) = \chi(g)$ is an isomorphism of topological groups

Example

Example A.6.3 (Self-Dual Groups).

- $\widehat{\mathbb{R}} \cong \mathbb{R}$: characters are $\chi_t(x) = e^{2\pi i t x}$
- $\widehat{\mathbb{T}} \cong \mathbb{Z}$: characters are $\chi_n(e^{i\theta}) = e^{in\theta}$
- $\widehat{\mathbb{Z}} \cong \mathbb{T}$: characters are $\chi_\theta(n) = e^{in\theta}$
- $\widehat{\mathbb{Z}/n\mathbb{Z}} \cong \mathbb{Z}/n\mathbb{Z}$: n-th roots of unity

The Fourier transform can be understood as the isomorphism between $L^2(G)$ and $L^2(\widehat{G})$. This perspective unifies classical Fourier series (for $\mathbb{T}$) and Fourier transforms (for $\mathbb{R}$).

★ ✦ ★

Greek Meditation

The Mirror of Duality: Plato's Forms and Their Reflections

In Plato's allegory of the cave, prisoners see only shadows—projections of true Forms existing in a higher realm. The dual space V^* is like this realm of shadows: each functional in V^* is a

"projection" that captures one aspect of vectors in V.

Yet here mathematics transcends philosophy: through the double dual V^{**}, we discover that the shadow can perfectly reflect the Form. For finite-dimensional spaces, $V \cong V^{**}$ canonically—the "shadows of shadows" recover the original reality.

Heraclitus taught that "the way up and the way down are one and the same." So too with duality: the path from V to V^* and back through V^{**} returns us home, transformed yet unchanged.

Duality reveals that to understand a structure, we must also understand what observes it. The observer and observed form a unified whole, each defining the other.

★

Zen Meditation

The Dual: Two That Are One, One That Is Two

What is the sound of one hand clapping?

The master asks: "Where is the dual space?"
The student points to a diagram: "Here is V, here is V^*."
The master smiles: "You see two. Look again."

A functional does not exist apart from what it measures. A vector does not exist apart from how it is measured. Subject and object arise together, like two sides of one coin.

The double dual teaches: going away and returning home are the same journey. The map $v \mapsto ((\varphi \mapsto \varphi(v)))$ shows that a thing IS its relationship to all possible observations of it.

In duality, we learn: emptiness is form, form is emptiness. The space and its dual interpenetrate completely. To truly understand one, you must become the other.

When you can hear with your eyes and see with your ears, you will understand Pontryagin duality.

✦

Practice Problems

Problem A.1. Let $V = \mathbb{R}^3$ with standard basis. Define $\varphi \in V^*$ by $\varphi(x, y, z) = 2x - y + 3z$. Express φ in terms of the dual basis.

> ***Solution.*** Solution. The dual basis consists of e_1^*, e_2^*, e_3^* where $e_i^*(x, y, z)$ extracts the i-th coordinate. We have:
>
> $$\varphi(x, y, z) = 2x - y + 3z = 2e_1^*(x, y, z) - e_2^*(x, y, z) + 3e_3^*(x, y, z)$$
>
> Therefore $\varphi = 2e_1^* - e_2^* + 3e_3^*$.

Problem A.2. Prove that the map $\iota : V \to V^{**}$ is natural (independent of basis choice).

> ***Solution.*** Solution. The definition $\iota(v)(\varphi) = \varphi(v)$ makes no reference to any basis. It depends only on the linear structures of V and V^*, and the evaluation of functionals. Any isomorphism $V \to V^{**}$ obtained by choosing bases would depend on those choices, but ι is defined purely in terms of the abstract vector space structure.

Problem A.3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation by 90° counterclockwise. Find the matrix of T^* with respect to the dual basis.

> ***Solution.*** Solution. The matrix of T is:
>
> $$[T] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
>
> The matrix of T^* is the transpose:
>
> $$[T^*] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
>
> This represents rotation by 90° clockwise (the inverse rotation), which makes sense: if T rotates vectors counterclockwise, then T^* rotates covectors clockwise to maintain the pairing.

Problem A.4. Let $W = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$. Find $W^\perp \subseteq (\mathbb{R}^3)^*$.

> ***Solution.*** Solution. We have $W = \ker(\varphi_0)$ where $\varphi_0(x, y, z) = x + y + z$. A functional $\varphi \in (\mathbb{R}^3)^*$ belongs to $W^\perp$ if and only if $\varphi(v) = 0$ for all $v \in W$.
>
> Writing $\varphi(x, y, z) = ax + by + cz$, the condition $\varphi \in W^\perp$ means $ax + by + cz = 0$ whenever $x + y + z = 0$. This forces φ to be a multiple of φ_0:
>
> $$W^\perp = \operatorname{span}\{\varphi_0\} = \{c(x + y + z) : c \in \mathbb{R}\}$$
>
> Note that $\dim(W) = 2$ and $\dim(W^\perp) = 1$, confirming $\dim(W) + \dim(W^\perp) = 3$.

Problem A.5. Show that if $T : V \to W$ is injective, then $T^* : W^* \to V^*$ is surjective.

Solution. Solution. Let $\varphi \in V^*$ be arbitrary. Since T is injective, $T(V)$ is a subspace of W isomorphic to V. We can define $\psi : T(V) \to F$ by $\psi(T(v)) = \varphi(v)$ (well-defined since T is injective). Extend ψ to all of W by choosing a complement and setting $\psi = 0$ on the complement. Then $T^*(\psi) = \varphi$, proving surjectivity.

Problem A.6. Compute $\widehat{\mathbb{R}/\mathbb{Z}}$ (the Pontryagin dual).

Solution. Solution. Since $\mathbb{R}/\mathbb{Z} \cong \mathbb{T}$ (the circle group), we have:

$$\widehat{\mathbb{R}/\mathbb{Z}} \cong \widehat{\mathbb{T}} \cong \mathbb{Z}$$

Explicitly, characters of $\mathbb{T}$ are given by $\chi_n(e^{i\theta}) = e^{in\theta}$ for $n \in \mathbb{Z}$.

Problem A.7. Prove that $(S \cap T)^\perp = S^\perp + T^\perp$ for subspaces $S, T \subseteq V$.

Solution. Solution. ($\supseteq$) If $\varphi \in S^\perp$, then $\varphi(v) = 0$ for all $v \in S$, hence for all $v \in S \cap T$. Similarly for $\psi \in T^\perp$. Thus $S^\perp + T^\perp \subseteq (S \cap T)^\perp$.

($\subseteq$) By dimension: $\dim(S^\perp) = n - \dim(S)$, $\dim(T^\perp) = n - \dim(T)$. Using inclusion-exclusion:

$$\begin{aligned}\dim(S^\perp + T^\perp) &= \dim(S^\perp) + \dim(T^\perp) - \dim(S^\perp \cap T^\perp) \\ &= 2n - \dim(S) - \dim(T) - \dim((S+T)^\perp) \\ &= 2n - \dim(S) - \dim(T) - (n - \dim(S+T)) \\ &= n - (\dim(S) + \dim(T) - \dim(S+T)) = n - \dim(S \cap T) = \dim((S \cap T)^\perp)\end{aligned}$$

Since we've shown $S^\perp + T^\perp \subseteq (S \cap T)^\perp$ and they have the same dimension, equality holds.

Problem A.8. Show that for a linear map $T : V \to W$, we have $\ker(T^*) = (\mathrm{Im}(T))^\perp$.

Solution. Solution. ($\subseteq$) If $\varphi \in \ker(T^*)$, then $T^*\varphi = 0$, so $(T^*\varphi)(v) = 0$ for all $v \in V$. By definition, $\varphi(T(v)) = 0$ for all v, meaning φ vanishes on $\mathrm{Im}(T)$. Thus $\varphi \in (\mathrm{Im}(T))^\perp$.

($\supseteq$) If $\varphi \in (\mathrm{Im}(T))^\perp$, then $\varphi(w) = 0$ for all $w \in \mathrm{Im}(T)$. For any $v \in V$, we have $T(v) \in \mathrm{Im}(T)$, so:

$$(T^*\varphi)(v) = \varphi(T(v)) = 0$$

Thus $T^*\varphi = 0$, i.e., $\varphi \in \ker(T^*)$.

Problem A.9. Prove that $V \cong V^*$ if and only if there exists a non-degenerate bilinear form on V.

Solution. Solution. ($\Rightarrow$) Given an isomorphism $\phi : V \to V^*$, define $B : V \times V \to F$ by:

$$B(v, w) = (\phi(v))(w)$$

This is bilinear and non-degenerate: if $B(v,w)=0$ for all w, then $\phi(v)=0$, so $v=0$ (since ϕ is injective).

($\Leftarrow$) Given a non-degenerate bilinear form B, define $\phi: V \to V^*$ by $\phi(v)(w) = B(v,w)$. Non-degeneracy ensures injectivity: if $\phi(v)=0$, then $B(v,w)=0$ for all w, forcing $v=0$. For finite-dimensional V, injectivity implies surjectivity.

Problem A.10 (Challenge) Let V be infinite-dimensional. Show that $\dim(V^*) > \dim(V)$.

Solution. Solution. Let $\{e_i : i \in I\}$ be a basis for V (possibly infinite). For each i, define $e_i^* \in V^*$ by $e_i^*(e_j) = \delta_{ij}$ (extend linearly). These are linearly independent: if $\sum_{i \in J} c_i e_i^* = 0$ (finite sum), evaluate at e_j to get $c_j = 0$.

Thus $|I| \leq \dim(V^*)$. But there are more functionals than just these coordinate functionals. For instance, if V is a normed space, bounded linear functionals form a strictly larger set. By cardinality arguments (using the axiom of choice), $\dim(V^*) = 2^{|I|} > |I| = \dim(V)$ for infinite I.

Problem A.11 Describe the dual of the polynomial ring $\mathbb{R}[x]$ as an $\mathbb{R}$-vector space.

Solution. Solution. $(\mathbb{R}[x])^* = \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}[x], \mathbb{R})$ is the space of all linear functionals on polynomials. Important examples include:

- Coefficient extraction: $\varphi_n(p) =$ coefficient of x^n in p
- Evaluation: $\delta_a(p) = p(a)$ for fixed $a \in \mathbb{R}$
- Integration: $\varphi(p) = \int_0^1 p(x)\,dx$

The dual basis to $\{1, x, x^2, \ldots\}$ consists of the coefficient functionals $\{\varphi_0, \varphi_1, \varphi_2, \ldots\}$. Thus $(\mathbb{R}[x])^* \cong \mathbb{R}^\infty$ (sequences), but note that not all linear functionals are represented by sequences (e.g., evaluation at non-integer powers isn't).

Problem A.12 Show that $\mathrm{rank}(T) = \mathrm{rank}(T^*)$ for any linear map $T: V \to W$ between finite-dimensional spaces.

Solution. Solution. From Problem A.8, $\ker(T^*) = (\mathrm{Im}(T))^\perp$. By the dimension formula:

$$\dim(\ker(T^*)) = \dim((\mathrm{Im}(T))^\perp) = \dim(W) - \dim(\mathrm{Im}(T)) = \dim(W) - \mathrm{rank}(T)$$

By rank-nullity for $T^*: W^* \to V^*$:

$$\mathrm{rank}(T^*) = \dim(W^*) - \dim(\ker(T^*)) = \dim(W) - (\dim(W) - \mathrm{rank}(T)) = \mathrm{rank}(T)$$

Problem A.13 Prove that for finite-dimensional V, the map $W \mapsto W^\perp$ is an order-reversing bijection between subspaces of V and subspaces of V^*.

Solution. Solution. *Order-reversing:* If $W_1 \subseteq W_2$, then every functional vanishing on W_2 also vanishes on W_1, so $W_2^\perp \subseteq W_1^\perp$.

Bijection: Define $\Phi : \{\text{subspaces of } V\} \to \{\text{subspaces of } V^*\}$ by $\Phi(W) = W^\perp$. The inverse is $\Psi(T) = T_\perp$ (using $V \cong V^{**}$). We've shown $(W^\perp)_\perp = W$, so $\Psi \circ \Phi = \mathrm{id}$. Similarly $\Phi \circ \Psi = \mathrm{id}$.

Problem A.14 Let $G = \mathbb{Z}_2 \times \mathbb{Z}_3$. Describe all characters $\chi : G \to \mathbb{T}$ explicitly.

Solution. Solution. By Pontryagin duality, $\widehat{\mathbb{Z}_2 \times \mathbb{Z}_3} \cong \widehat{\mathbb{Z}_2} \times \widehat{\mathbb{Z}_3} \cong \mathbb{Z}_2 \times \mathbb{Z}_3$.

Explicitly, characters are determined by their values on generators:

- $\chi(1, 0)$ must be a square root of unity: $\chi(1,0) \in \{1, -1\}$
- $\chi(0, 1)$ must be a cube root of unity: $\chi(0,1) \in \{1, e^{2\pi i/3}, e^{4\pi i/3}\}$

There are $2 \times 3 = 6$ characters total, labeled by pairs $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_3$:

$$\chi_{(a,b)}(m, n) = (-1)^{am} \cdot e^{2\pi i bn/3}$$

Problem A.15 (Challenge) Prove that a finitely generated module M over a commutative ring R is projective if and only if M^* is flat.

Solution. Solution. (Sketch) Recall:

- M projective means: every surjection $N \twoheadrightarrow M$ splits
- M^* flat means: tensoring with M^* preserves exactness

($\Rightarrow$) If M is projective, then M is a direct summand of a free module: $F = M \oplus N$. Thus $F^* = M^* \oplus N^*$. Since F^* is free (hence flat), and direct summands of flat modules are flat, M^* is flat.

($\Leftarrow$) If M^* is flat, consider the evaluation map $\epsilon : M \to M^{**}$. Flatness of M^* ensures that certain exactness properties transfer. For finitely generated modules, this forces M to be a direct summand of a finite free module, hence projective. (Full proof requires detailed homological algebra.)

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End of Part A

Part B: Lie Groups and Lie Algebras

PART B

✦

LIE GROUPS & LIE ALGEBRAS

The theory of Lie groups stands at the intersection of algebra, analysis, and geometry, providing a unified framework for understanding continuous symmetries. Named after the Norwegian mathematician Sophus Lie, these structures combine the algebraic properties of groups with the geometric smoothness of manifolds.

Where discrete groups capture digital symmetries (like reflections and rotations of a cube), Lie groups describe analog symmetries—continuous transformations that can be smoothly interpolated. Every rotation of a sphere, every translation in space, every Lorentz transformation in relativity is an element of a Lie group.

The bridge between the infinitesimal and the global is provided by Lie algebras: linearizations that capture the group's local structure. Through the exponential map, we can reconstruct much of the global group from its tangent space at the identity. This part explores this remarkable connection and its profound implications.

★ ✦ ★

B.1 Introduction to Lie Groups

A Lie group seamlessly merges two fundamental structures: the algebraic framework of groups and the geometric structure of smooth manifolds.

> **Definition**
>
> Definition B.1.1 (Lie Group).
> A Lie group is a smooth manifold G equipped with a group structure such that the group operations are smooth maps:
>
> - Multiplication: $\mu : G \times G \to G$, $(g, h) \mapsto gh$
> - Inversion: $\iota : G \to G$, $g \mapsto g^{-1}$
>
> are both C^∞ (infinitely differentiable).

The dimension of a Lie group is its dimension as a manifold. The identity element is denoted e (or sometimes 1 or id).

Example

Example B.1.2 (The General Linear Group).
The general linear group $\mathrm{GL}(n, \mathbb{R})$ consists of all invertible $n \times n$ real matrices:

$$\mathrm{GL}(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det(A) \neq 0\}$$

This is an open subset of $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ (where $\det \neq 0$), hence a smooth manifold of dimension n^2. Matrix multiplication and inversion (via Cramer's rule) are smooth, making $\mathrm{GL}(n, \mathbb{R})$ a Lie group.

Example

Example B.1.3 (The Special Linear Group).
The special linear group is:

$$\mathrm{SL}(n, \mathbb{R}) = \{A \in \mathrm{GL}(n, \mathbb{R}) : \det(A) = 1\}$$

This is a closed submanifold of $\mathrm{GL}(n, \mathbb{R})$ defined by the smooth equation $\det(A) = 1$. Since $d(\det)|_I : M_n(\mathbb{R}) \to \mathbb{R}$ given by $A \mapsto \mathrm{tr}(A)$ is surjective, the implicit function theorem shows $\mathrm{SL}(n, \mathbb{R})$ is a smooth submanifold of dimension $n^2 - 1$.

Example

Example B.1.4 (Orthogonal and Special Orthogonal Groups).
The orthogonal group preserves the Euclidean inner product:

$$\mathrm{O}(n) = \{A \in \mathrm{GL}(n, \mathbb{R}) : A^T A = I\}$$

The condition $A^T A = I$ imposes $\frac{n(n+1)}{2}$ independent constraints (from the symmetric matrix $A^T A$), giving:

$$\dim(\mathrm{O}(n)) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$$

The special orthogonal group $\mathrm{SO}(n) = \mathrm{O}(n) \cap \mathrm{SL}(n, \mathbb{R})$ consists of rotations (orientation-preserving orthogonal transformations). Since $\det(A) = \pm 1$ for $A \in \mathrm{O}(n)$, we have $\mathrm{O}(n) = \mathrm{SO}(n) \sqcup \mathrm{SO}(n) \cdot R$ where R is any reflection. Thus $\mathrm{SO}(n)$ is the connected component of the identity in $\mathrm{O}(n)$, with the same dimension.

Example

Example B.1.5 (Unitary and Special Unitary Groups).
Over $\mathbb{C}$, the unitary group preserves the Hermitian inner product:

$$\mathrm{U}(n) = \{A \in \mathrm{GL}(n, \mathbb{C}) : A^* A = I\}$$

where $A^* = \overline{A}^T$ is the conjugate transpose. This is a compact Lie group of dimension n^2.

The special unitary group is:

$$\mathrm{SU}(n) = \{A \in \mathrm{U}(n) : \det(A) = 1\}$$

Since $|\det(A)| = 1$ for $A \in \mathrm{U}(n)$ and we require $\det(A) = 1$, this imposes one real constraint, giving $\dim(\mathrm{SU}(n)) = n^2 - 1$.

Example

Example B.1.6 (The Symplectic Group).
The symplectic group preserves a non-degenerate skew-symmetric bilinear form:

$$\mathrm{Sp}(2n, \mathbb{R}) = \left\{A \in \mathrm{GL}(2n, \mathbb{R}) : A^T J A = J\right\}$$

where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. This group arises naturally in classical mechanics (phase space symmetries) and has dimension $\dim(\mathrm{Sp}(2n, \mathbb{R})) = n(2n+1)$.

Example

Example B.1.7 (The Circle Group).
The circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ with complex multiplication is a compact abelian Lie group of dimension 1. It can be identified with $\mathrm{U}(1) = \{e^{i\theta} : \theta \in \mathbb{R}\}$ or with $\mathrm{SO}(2)$ (rotations of the plane).

Matrix Lie Groups

All the examples above are matrix Lie groups—closed subgroups of $\mathrm{GL}(n, \mathbb{C})$ for some n. A fundamental theorem states that every closed subgroup of a Lie group is itself a Lie subgroup (smooth submanifold with induced Lie group structure).

Theorem

Theorem B.1.8 (Cartan's Closed Subgroup Theorem).
Every closed subgroup of a Lie group is a Lie subgroup (i.e., a smooth submanifold).

This theorem is powerful because it means we need only verify the algebraic (group) and topological (closed) properties; smoothness comes automatically.

Topological Properties

Lie groups enjoy special topological properties:

> **Proposition**
>
> Proposition B.1.9.
>
> 1. Every Lie group is locally compact and Hausdorff
> 2. A compact matrix Lie group has finitely many connected components
> 3. Every connected Lie group is path-connected
> 4. The connected component G_0 containing the identity is a normal subgroup

✦

B.2 The Exponential Map

The exponential map is the bridge between a Lie group and its Lie algebra, connecting infinitesimal generators to finite transformations.

> **Definition**
>
> Definition B.2.1 (One-Parameter Subgroup).
> A one-parameter subgroup of a Lie group G is a smooth group homomorphism $\gamma : \mathbb{R} \to G$. That is:
>
> - $\gamma(s+t) = \gamma(s)\gamma(t)$ for all $s, t \in \mathbb{R}$
> - γ is a smooth map

One-parameter subgroups are the Lie group analogue of "lines through the origin" in vector spaces.

> **Theorem**
>
> Theorem B.2.2 (One-Parameter Subgroups and Tangent Vectors).
> There is a bijection between one-parameter subgroups of G and tangent vectors at the identity $e \in G$. Given a tangent vector $X \in T_eG$, there exists a unique one-parameter subgroup $\gamma_X : \mathbb{R} \to G$ with $\gamma_X(0) = e$ and $\gamma_X'(0) = X$.

Definition

Definition B.2.3 (Exponential Map).
The exponential map $\exp : T_eG \to G$ is defined by:

$$\exp(X) = \gamma_X(1)$$

where γ_X is the unique one-parameter subgroup with initial velocity X.

For matrix Lie groups, this coincides with the matrix exponential:

Example

Example B.2.4 (Matrix Exponential).
For $G \subseteq \mathrm{GL}(n, \mathbb{C})$ a matrix Lie group, the exponential map is the matrix exponential:

$$\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

This series converges for all matrices $A \in M_n(\mathbb{C})$ and satisfies:

- $\exp(0) = I$
- $\exp((s+t)A) = \exp(sA)\exp(tA)$
- $\exp(A)^{-1} = \exp(-A)$
- If $AB = BA$, then $\exp(A+B) = \exp(A)\exp(B)$

Example

Example B.2.5 (Exponential for $\mathrm{SO}(2)$).
The Lie algebra of $\mathrm{SO}(2)$ consists of 2×2 skew-symmetric matrices:

$$\mathfrak{so}(2) = \left\{ \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} : \theta \in \mathbb{R} \right\}$$

The exponential map is:

$$\exp \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

This is surjective: every rotation comes from exponentiating a skew-symmetric matrix.

Example

Example B.2.6 (Exponential for $\mathrm{SO}(3)$).
For $\mathrm{SO}(3)$, every rotation can be described by a rotation axis (unit vector $\mathbf{n}$) and angle θ. The exponential map takes a skew-symmetric matrix to a rotation:

$$\exp(\theta \mathbf{n}^\wedge) = R_{\mathbf{n}}(\theta)$$

where $\mathbf{n}^\wedge$ is the skew-symmetric matrix corresponding to $\mathbf{n}$ via the cross product: $\mathbf{n}^\wedge \mathbf{v} = \mathbf{n} \times \mathbf{v}$.

Theorem

Theorem B.2.7 (Properties of the Exponential Map).
Let G be a Lie group with exponential map $\exp : \mathfrak{g} \to G$. Then:

1. $\exp(0) = e$
2. $\exp(tX) = \gamma_X(t)$ for all $t \in \mathbb{R}$
3. $d(\exp)|_0 : \mathfrak{g} \to T_e G$ is the identity map
4. $\exp$ is a local diffeomorphism near $0 \in \mathfrak{g}$
5. For abelian G: $\exp(X + Y) = \exp(X)\exp(Y)$
6. General case: $\exp(X)\exp(Y) = \exp(X + Y + \frac{1}{2}[X, Y] + \cdots)$ (Baker-Campbell-Hausdorff formula)

Theorem

Theorem B.2.8 (Baker-Campbell-Hausdorff Formula).
For X, Y in a Lie algebra $\mathfrak{g}$ with $\|X\|, \|Y\|$ sufficiently small:

$$\log(\exp(X)\exp(Y)) = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [Y, X]] + \cdots$$

The right-hand side is a (convergent) infinite series involving nested commutators.

This formula shows that the group multiplication near the identity is completely determined by the Lie bracket.

✦

B.3 Lie Algebras

The tangent space at the identity of a Lie group inherits a natural algebraic structure from the group operations.

Definition

Definition B.3.1 (Lie Algebra).
A Lie algebra over a field $\mathbb{F}$ is a vector space $\mathfrak{g}$ equipped with a bilinear operation $[\cdot,\cdot]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ (called the Lie bracket) satisfying:

1. **Antisymmetry:** $[X,Y]=-[Y,X]$ for all $X,Y\in\mathfrak{g}$
2. **Jacobi identity:** $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$ for all $X,Y,Z\in\mathfrak{g}$

The Jacobi identity can be remembered mnemonically as "cyclic sum of nested brackets equals zero."

Definition

Definition B.3.2 (Lie Algebra of a Lie Group).
The Lie algebra of a Lie group G, denoted $\mathfrak{g}$ or $\mathrm{Lie}(G)$, is the tangent space T_eG at the identity, equipped with the Lie bracket:

$$[X,Y]=\frac{d^2}{ds\,dt}\Big|_{s=t=0}\exp(sX)\exp(tY)\exp(-sX)\exp(-tY)$$

For matrix Lie groups, this reduces to the commutator:

$$[X,Y]=XY-YX$$

Example

Example B.3.3 (Lie Algebra $\mathfrak{gl}(n,\mathbb{R})$).
The Lie algebra of $\mathrm{GL}(n,\mathbb{R})$ is:

$$\mathfrak{gl}(n,\mathbb{R})=M_n(\mathbb{R})$$

(all $n\times n$ real matrices) with Lie bracket $[A,B]=AB-BA$ (matrix commutator). This has dimension n^2.

Example

Example B.3.4 (Lie Algebra $\mathfrak{sl}(n,\mathbb{R})$).
The Lie algebra of $\mathrm{SL}(n,\mathbb{R})$ is:

$$\mathfrak{sl}(n,\mathbb{R})=\{A\in M_n(\mathbb{R}):\mathrm{tr}(A)=0\}$$

(traceless matrices). Since $\mathrm{tr}([A,B])=\mathrm{tr}(AB-BA)=0$, this is closed under the commutator. Dimension: n^2-1.

Example

Example B.3.5 (Lie Algebra $\mathfrak{so}(n)$).
The Lie algebra of $\mathrm{SO}(n)$ is:

$$\mathfrak{so}(n) = \{A \in M_n(\mathbb{R}) : A^T = -A\}$$

(skew-symmetric matrices). If $A^T = -A$ and $B^T = -B$, then:

$$[A,B]^T = (AB - BA)^T = B^T A^T - A^T B^T = BA - AB = -[A,B]$$

so the commutator is skew-symmetric. Dimension: $\frac{n(n-1)}{2}$.

Example

Example B.3.6 (Lie Algebra $\mathfrak{su}(n)$).
The Lie algebra of $\mathrm{SU}(n)$ is:

$$\mathfrak{su}(n) = \{A \in M_n(\mathbb{C}) : A^* = -A, \operatorname{tr}(A) = 0\}$$

(skew-Hermitian traceless matrices). Dimension: $n^2 - 1$.

Lie Algebra Homomorphisms

Definition

Definition B.3.7 (Lie Algebra Homomorphism).
A Lie algebra homomorphism is a linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ that preserves the bracket:

$$\phi([X,Y]_{\mathfrak{g}}) = [\phi(X), \phi(Y)]_{\mathfrak{h}}$$

Theorem

Theorem B.3.8 (Lie Group Homomorphisms Induce Lie Algebra Homomorphisms).
If $\Phi : G \to H$ is a Lie group homomorphism, then the differential $d\Phi_e : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism. Moreover, the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{d\Phi_e} & \mathfrak{h} \\ \downarrow \exp_G & & \downarrow \exp_H \\ G & \xrightarrow{\Phi} & H \end{array}$$

✦

B.4 Structure Theory

The structure theory of Lie algebras provides a classification of simple Lie algebras, one of the great achievements of 19th and 20th century mathematics.

Ideals and Simple Lie Algebras

Definition

Definition B.4.1 (Ideal).
A subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is an ideal if $[X, Y] \in \mathfrak{h}$ for all $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$.

Definition

Definition B.4.2 (Simple Lie Algebra).
A Lie algebra $\mathfrak{g}$ is simple if:

1. $\dim(\mathfrak{g}) > 1$
2. $\mathfrak{g}$ is non-abelian (i.e., $[\mathfrak{g}, \mathfrak{g}] \neq 0$)
3. $\mathfrak{g}$ has no proper non-trivial ideals

The Killing Form

The Killing form is a bilinear form that encodes essential structural information about a Lie algebra.

Definition

Definition B.4.3 (Adjoint Representation).
The adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is defined by:

$$\mathrm{ad}_X(Y) = [X, Y]$$

This is a Lie algebra homomorphism: $\mathrm{ad}_{[X,Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y]$.

Definition

Definition B.4.4 (Killing Form).
The Killing form is the symmetric bilinear form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ defined by:

$$\kappa(X, Y) = \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y)$$

Theorem

Theorem B.4.5 (Cartan's Criterion).
A Lie algebra $\mathfrak{g}$ over a field of characteristic 0 is semisimple if and only if its Killing form is non-degenerate.

Cartan Subalgebras and Root Systems

Definition

Definition B.4.6 (Cartan Subalgebra).
A Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is a maximal abelian subalgebra consisting of semisimple elements (diagonalizable under the adjoint representation).

For a semisimple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$, we can decompose $\mathfrak{g}$ into weight spaces:

Theorem

Theorem B.4.7 (Root Space Decomposition).
Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ with Cartan subalgebra $\mathfrak{h}$. Then:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

where $\Phi \subseteq \mathfrak{h}^*$ (the root system) and:

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}$$

Definition

Definition B.4.8 (Root System).
A root system in a real vector space E is a finite set $\Phi \subseteq E$ of nonzero vectors (called roots) satisfying:

1. Φ spans E
2. For each $\alpha \in \Phi$, the reflection s_α (through the hyperplane perpendicular to α) maps Φ to itself
3. For $\alpha, \beta \in \Phi$: $\frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$
4. If $\alpha \in \Phi$, then the only multiples of α in Φ are $\pm\alpha$

Dynkin Diagrams and Classification

Root systems can be encoded in Dynkin diagrams—graphs that capture the angles and length ratios between simple roots.

Theorem

Theorem B.4.9 (Classification of Simple Lie Algebras).
Every complex simple Lie algebra is isomorphic to exactly one of the following:

- A_n **(**$n \geq 1$**):** $\mathfrak{sl}(n+1, \mathbb{C})$ — dimension $(n+1)^2 - 1$
- B_n **(**$n \geq 2$**):** $\mathfrak{so}(2n+1, \mathbb{C})$ — dimension $n(2n+1)$
- C_n **(**$n \geq 3$**):** $\mathfrak{sp}(2n, \mathbb{C})$ — dimension $n(2n+1)$
- D_n **(**$n \geq 4$**):** $\mathfrak{so}(2n, \mathbb{C})$ — dimension $n(2n-1)$
- **Exceptional algebras:** E_6, E_7, E_8, F_4, G_2 — dimensions 78, 133, 248, 52, 14 respectively

The Dynkin diagrams are:

Example

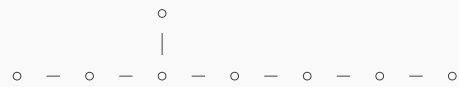
Example B.4.10 (Dynkin Diagram A_n).

$$\circ - \circ - \circ - \cdots - \circ - \circ$$

(Linear chain of n nodes)

Example

Example B.4.11 (Dynkin Diagram E_8).



The exceptional Lie algebra E_8 has dimension 248 and plays a role in string theory and moonshine.

✦

B.5 Representation Theory

Representations make abstract Lie groups and algebras concrete by realizing them as symmetries of vector spaces.

Definition

Definition B.5.1 (Representation of a Lie Group).
A representation of a Lie group G on a vector space V is a group homomorphism:

$$\rho : G \to \mathrm{GL}(V)$$

that is smooth (as a map between manifolds). The dimension of V is called the dimension of the representation.

Definition

Definition B.5.2 (Representation of a Lie Algebra).
A representation of a Lie algebra $\mathfrak{g}$ on a vector space V is a Lie algebra homomorphism:

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V)$$

where $\mathfrak{gl}(V) = \mathrm{End}(V)$ with commutator bracket.

Example

Example B.5.3 (Defining Representation).
For matrix Lie groups $G \subseteq \mathrm{GL}(n, \mathbb{F})$, the defining representation is the inclusion $G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ (the representation that defines G as matrices).

Example

Example B.5.4 (Adjoint Representation of a Lie Group).
For a Lie group G, the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is defined by:

$$\mathrm{Ad}_g(X) = \frac{d}{dt}\Big|_{t=0} g \exp(tX) g^{-1}$$

This measures how conjugation by g acts on the Lie algebra.

Weights and Weight Spaces

Definition

Definition B.5.5 (Weight).
Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation with Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$. A weight is a linear functional $\lambda \in \mathfrak{h}^*$ such that the weight space:

$$V_\lambda = \{v \in V : \rho(H)v = \lambda(H)v \text{ for all } H \in \mathfrak{h}\}$$

is nonzero.

Theorem

Theorem B.5.6 (Weight Space Decomposition).
For a finite-dimensional representation of a semisimple Lie algebra:

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$$

Highest Weight Theory

Theorem

Theorem B.5.7 (Highest Weight Theorem).
Irreducible finite-dimensional representations of a semisimple Lie algebra $\mathfrak{g}$ are in bijection with dominant integral weights. Each irreducible representation has a unique (up to scaling) highest weight vector v_λ satisfying:

- $H \cdot v_\lambda = \lambda(H) v_\lambda$ for all $H \in \mathfrak{h}$
- $E_\alpha \cdot v_\lambda = 0$ for all positive roots α

Example

Example B.5.8 (Representations of $\mathfrak{sl}(2, \mathbb{C})$).
The irreducible finite-dimensional representations of $\mathfrak{sl}(2, \mathbb{C})$ are labeled by non-negative integers $n \in \mathbb{Z}_{\geq 0}$, with dimension $n + 1$. These correspond to symmetric powers:

$$V_n = \mathrm{Sym}^n(\mathbb{C}^2)$$

The $n = 1$ representation is the defining representation $\mathbb{C}^2$. The $n = 2$ representation is $\mathrm{Sym}^2(\mathbb{C}^2) \cong \mathbb{C}^3$.

✦

Greek Meditation

The Eternal Dance: Symmetry in Motion

The pre-Socratic philosopher Heraclitus proclaimed: "Everything flows, nothing stands still." Yet within this flux, patterns persist—the logos, the underlying order.

Lie groups embody this paradox: continuous transformation that preserves structure. A rotating sphere changes at every moment, yet remains perfectly spherical. The transformation and the

invariant interpenetrate.

The exponential map reveals profound wisdom: the infinite can be understood through the infinitesimal. To know all rotations of 3-space, study only the tangent space at the identity—a mere 3-dimensional linear space. From the seed, the entire tree grows.

Plato taught that geometric forms possess eternal reality beyond their physical manifestations. The Lie group $\mathrm{SO}(3)$ is the Platonic Form of Rotation itself—not any particular rotation, but the essence of all rotations. Its Lie algebra $\mathfrak{so}(3)$ is the form of infinitesimal rotation.

The classification of simple Lie algebras—$A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$—reveals a hidden music in the cosmos. Like the Pythagorean discovery that musical harmonies arise from simple ratios, we find that all continuous symmetries arise from these exceptional forms.

E_8, the most beautiful and mysterious, has 248 dimensions yet is completely determined by its root system—a finite crystallographic pattern in 8-dimensional space. Here we see the One and the Many reconciled: infinite complexity arising from finite essence.

★

Zen Meditation

The Still Point of the Turning World

A student asked the master: "What is the identity element?"
The master replied: "The place where you have never left."

All transformations emerge from stillness. The identity e is not merely one element among many—it is the source and destination of all one-parameter subgroups. Every motion begins and ends in rest.

The Lie algebra is the mind of the Lie group. As thoughts arise in awareness, group elements arise from the tangent space. The exponential map is like breathing: $\exp(X)$ is the exhalation, $\log(g)$ is the inhalation. Thought becomes action, action reveals thought.

Consider the Killing form $\kappa(X, Y) = \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y)$. It measures how X and Y "interact" through their adjoint actions. Non-degeneracy means no element acts trivially—each carries unique information about the whole. This is interdependence: every part contains the whole.

The Baker-Campbell-Hausdorff formula teaches:
$\exp(X)\exp(Y) = \exp(X + Y + \frac{1}{2}[X, Y] + \cdots)$. Two actions, performed in sequence, equal a single action—but not merely their sum. The commutator $[X, Y]$ captures the "interference" between non-commuting actions. This is the sound of two hands clapping.

> When you understand that $\mathrm{SO}(3)$ and $\mathfrak{so}(3)$ are not two things but two views of one reality, you will understand Dogen's teaching: "To study the Way is to study the self. To study the self is to forget the self."
>
> The group sees globally. The algebra sees locally. Neither is complete without the other.
>
> *What is your original face before you were born? It is* $\exp(t\mathfrak{g})|_{t=0}$.

B.6 Practice Problems

Problem B.1. Show that $\mathrm{SL}(2,\mathbb{R})$ is a Lie group by verifying that multiplication and inversion are smooth maps.

Problem B.2. Prove that $\mathrm{SO}(n)$ is a compact Lie group for all $n \geq 1$.

Problem B.3. Compute the dimension of $\mathrm{U}(n)$ directly by counting independent parameters.

Problem B.4. Show that the matrix exponential of a skew-symmetric real matrix is always orthogonal: if $A^T = -A$, then $\exp(A) \in \mathrm{O}(n)$.

Problem B.5. Compute $\exp\begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$ explicitly and verify it gives a rotation.

Problem B.6. Find the Lie algebra of the Heisenberg group:

$$H = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$$

Problem B.7. Verify that the Lie bracket $[X, Y] = XY - YX$ satisfies the Jacobi identity.

Problem B.8. Show that $\mathfrak{su}(2) \cong \mathfrak{so}(3)$ as Lie algebras (isomorphic but not equal).

Problem B.9. Prove that every Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ maps ideals to ideals.

Problem B.10. Compute the Killing form for $\mathfrak{sl}(2,\mathbb{R})$.

Problem B.11. Show that the center of a Lie algebra $\mathfrak{g}$ is $Z(\mathfrak{g}) = \{X \in \mathfrak{g} : [X, Y] = 0 \text{ for all } Y \in \mathfrak{g}\}$.

Problem B.12. Find a basis for $\mathfrak{sl}(2,\mathbb{C})$ as a real Lie algebra and determine its dimension.

Problem B.13. Let $\mathfrak{g}$ be the Lie algebra of $\mathrm{SU}(2)$ with standard basis:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

compute $[H, E]$, $[H, F]$, and $[E, F]$.

Problem B.13. Show that $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra automorphism for each $g \in G$.

Problem B.14. Prove that $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ is a derivation: $\mathrm{ad}_X([Y, Z]) = [\mathrm{ad}_X(Y), Z] + [Y, \mathrm{ad}_X(Z)]$.

Problem B.15. Describe all 2-dimensional Lie algebras over $\mathbb{R}$ up to isomorphism.

Problem B.16. Show that the exponential map $\exp : \mathfrak{so}(3) \to \mathrm{SO}(3)$ is surjective but not injective.

Problem B.17. Prove that if G is abelian, then $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$.

Problem B.18. (Challenge) Verify the Baker-Campbell-Hausdorff formula up to second-order terms: show that $\exp(X)\exp(Y) = \exp(X + Y + \frac{1}{2}[X, Y] + O(3))$.

Problem B.19. Describe the root system of $\mathfrak{sl}(3, \mathbb{C})$.

Problem B.20. (Challenge) Show that $\mathrm{SU}(2)$ is diffeomorphic to the 3-sphere S^3.

B.7 Solutions

> ***Solution.*** Solution B.1. $\mathrm{SL}(2, \mathbb{R}) = \{A \in M_2(\mathbb{R}) : \det(A) = 1\}$ is a smooth manifold (level set of smooth function with regular value). Matrix multiplication is polynomial in entries, hence smooth. Inversion uses Cramer's rule: $A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A)$, which is smooth on $\mathrm{SL}(2, \mathbb{R})$ since $\det(A) = 1 \neq 0$.

> ***Solution.*** Solution B.2. $\mathrm{SO}(n) \subseteq \mathrm{O}(n)$ is closed (det = 1) in the compact set $\mathrm{O}(n) = \{A : A^T A = I\}$. Compactness of $\mathrm{O}(n)$ follows from $\|A\|_{\mathrm{op}} = 1$ for $A \in \mathrm{O}(n)$ (bounded) and closedness in $M_n(\mathbb{R})$. Heine-Borel implies compactness. Closed subset of compact is compact.

> ***Solution.*** Solution B.3. A unitary matrix $A \in \mathrm{U}(n)$ satisfies $A^* A = I$. Writing $A = X + iY$ with X, Y real, the condition $A^* A = I$ gives $(X - iY)(X + iY) = I$, or $X^T X + Y^T Y = I$ and $X^T Y = Y^T X$. These are $\frac{n(n+1)}{2} + \frac{n(n-1)}{2} = n^2$ independent real equations in $2n^2$ real variables, giving dimension $2n^2 - n^2 = n^2$.

> ***Solution.*** Solution B.4. If $A^T = -A$, then:
>
> $$\exp(A)^T \exp(A) = \exp(A^T)\exp(A) = \exp(-A)\exp(A) = \exp(-A + A) = \exp(0) = I$$
>
> (using A and $-A$ commute). Thus $\exp(A) \in \mathrm{O}(n)$. Moreover, $\det(\exp(A)) = e^{\mathrm{tr}(A)} = e^0 = 1$ (since $\mathrm{tr}(A) = 0$ for skew-symmetric A), so $\exp(A) \in \mathrm{SO}(n)$.

> ***Solution.*** Solution B.5. Let $A = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$. Then $A^2 = -\theta^2 I$, so:

$$\exp(A) = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots = I + A\left(1 - \frac{\theta^2}{3!} + \cdots\right) + A^2\left(\frac{1}{2!} - \frac{\theta^2}{4!} + \cdots\right)$$
$$= I + A\frac{\sin\theta}{\theta} - \theta^2 I\frac{1-\cos\theta}{\theta^2} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

This is indeed a rotation by θ.

Solution. Solution B.6. The Lie algebra $\mathfrak{h}$ consists of matrices $X = \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}$. A basis is:

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Computing: $[X, Y] = XY - YX = Z$, $[X, Z] = [Y, Z] = 0$. This is the Heisenberg algebra.

Solution. Solution B.7. We must verify $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$. Expanding:

$$[X,[Y,Z]] = X(YZ - ZY) - (YZ - ZY)X = XYZ - XZY - YZX + ZYX$$

Similarly for the other two terms. Summing all three gives:

$$(XYZ - XZY - YZX + ZYX) + (YZX - YXZ - ZXY + XZY) + (ZXY - ZYX -$$

All terms cancel in pairs.

Solution. Solution B.8. Both have dimension 3. Define $\phi : \mathfrak{su}(2) \to \mathfrak{so}(3)$ using the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

A basis for $\mathfrak{su}(2)$ is $\{i\sigma_1/2, i\sigma_2/2, i\sigma_3/2\}$. The map sends these to rotations about the x, y, z axes. The commutation relations $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$ match those of $\mathfrak{so}(3)$.

Solution. Solution B.9. Let $\mathfrak{i} \subseteq \mathfrak{g}$ be an ideal and $X \in \mathfrak{i}$. For any $Y \in \mathfrak{g}$ and $Z \in \mathfrak{h}$:

$$[\phi(Y), \phi(X)] = \phi([Y, X]) \in \phi(\mathfrak{i})$$

since $[Y, X] \in \mathfrak{i}$ by the ideal property. Thus $\phi(\mathfrak{i})$ is an ideal in $\mathfrak{h}$.

Solution. Solution B.10. Basis for $\mathfrak{sl}(2,\mathbb{R})$: $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

Compute $\mathrm{ad}_H(H) = 0$, $\mathrm{ad}_H(E) = 2E$, $\mathrm{ad}_H(F) = -2F$. Matrix of ad_H is $\mathrm{diag}(0, 2, -2)$. Similarly compute ad_E and ad_F. Then:

$$\kappa(H, H) = \mathrm{tr}(\mathrm{ad}_H^2) = 0 + 4 + 4 = 8$$
$$\kappa(E, F) = \mathrm{tr}(\mathrm{ad}_E \circ \mathrm{ad}_F) = 4$$

The Killing form is $\kappa = 8H \otimes H + 4(E \otimes F + F \otimes E)$, which is non-degenerate.

Solution. Solution B.11. By definition, $X \in Z(\mathfrak{g})$ iff $[X, Y] = 0$ for all $Y \in \mathfrak{g}$. This is a subspace (if $[X, Y] = [X', Y] = 0$ for all Y, then $[aX + bX', Y] = a[X, Y] + b[X', Y] = 0$). Clearly $[Z(\mathfrak{g}), \mathfrak{g}] = 0$.

Solution. Solution B.12. Direct computation:

$$[H, E] = HE - EH = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$$

$$[H, F] = -2F$$

$$[E, F] = H$$

These are the standard $\mathfrak{sl}(2, \mathbb{C})$ relations.

Solution. Solution B.13. Ad_g is linear by definition. We must show it preserves the bracket:

$$\mathrm{Ad}_g([X, Y]) = g[X, Y]g^{-1} = g(XY - YX)g^{-1} = gXg^{-1} \cdot gYg^{-1} - gYg^{-1} \cdot gXg^{-1}$$

$$= [\mathrm{Ad}_g(X), \mathrm{Ad}_g(Y)]$$

Solution. Solution B.14.

$$\mathrm{ad}_X([Y, Z]) = [X, [Y, Z]]$$

By Jacobi: $[X, [Y, Z]] = -[Y, [Z, X]] - [Z, [X, Y]] = [Y, [X, Z]] + [[X, Y], Z]$

$$= [\mathrm{ad}_X(Y), Z] + [Y, \mathrm{ad}_X(Z)]$$

This is the derivation property.

Solution. Solution B.15. There are two: (1) $\mathbb{R}^2$ with trivial bracket $[X, Y] = 0$ (abelian), (2) the non-abelian 2D algebra with basis $\{X, Y\}$ and $[X, Y] = Y$ (the "affine" algebra, Lie algebra of $\mathrm{Aff}(\mathbb{R})$).

Solution. Solution B.16. **Surjective:** Every rotation can be represented as a rotation about some axis by some angle $\theta \in [0, \pi]$. This comes from exponentiating the corresponding skew-symmetric matrix.

Not injective: Rotation by 2π about any axis gives the identity, but comes from exponentiating non-zero matrices. Thus $\exp(\theta \mathbf{n}^\wedge) = \exp((\theta + 2\pi k)\mathbf{n}^\wedge)$ for integer k.

Solution. Solution B.17. If G is abelian, then $\exp(X)\exp(Y) = \exp(Y)\exp(X)$ for all $X, Y \in \mathfrak{g}$. By Baker-Campbell-Hausdorff:

$$\exp(X + Y + \frac{1}{2}[X, Y] + \cdots) = \exp(Y + X + \frac{1}{2}[Y, X] + \cdots)$$

$$= \exp(X + Y - \frac{1}{2}[X, Y] + \cdots)$$

Comparing coefficients: $[X, Y] = 0$.

Solution. Solution B.18. Write $\exp(X) = I + X + \frac{X^2}{2} + O(3)$ and $\exp(Y) = I + Y + \frac{Y^2}{2} + O(3)$. Then:

$$\exp(X)\exp(Y) = (I + X + \frac{X^2}{2})(I + Y + \frac{Y^2}{2}) + O(3)$$

$$= I + X + Y + XY + \frac{X^2}{2} + \frac{Y^2}{2} + O(3)$$

$$= I + (X+Y) + \frac{(X+Y)^2}{2} + \frac{XY - YX}{2} + O(3)$$

$$= I + (X+Y) + \frac{(X+Y)^2}{2} + \frac{[X,Y]}{2} + O(3) = \exp(X + Y + \frac{[X,Y]}{2}) + O(3)$$

Solution. Solution B.19. Cartan subalgebra: $\mathfrak{h} = \{\mathrm{diag}(a_1, a_2, a_3) : a_1 + a_2 + a_3 = 0\}$ (dimension 2).
Roots: $\alpha_{ij}(H) = a_i - a_j$ for $i \neq j$. There are 6 roots: $\{\alpha_{12}, \alpha_{21}, \alpha_{13}, \alpha_{31}, \alpha_{23}, \alpha_{32}\}$, forming the root system of type A_2 (hexagonal pattern in plane).

Solution. Solution B.20. Parametrize $\mathrm{SU}(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : |a|^2 + |b|^2 = 1 \right\}$. Write $a = a_1 + ia_2$, $b = b_1 + ib_2$ with $a_i, b_i \in \mathbb{R}$. The condition $|a|^2 + |b|^2 = 1$ becomes:

$$a_1^2 + a_2^2 + b_1^2 + b_2^2 = 1$$

This is the unit 3-sphere $S^3 \subseteq \mathbb{R}^4$. The map $(a_1, a_2, b_1, b_2) \mapsto \begin{pmatrix} a_1 + ia_2 & b_1 + ib_2 \\ -(b_1 - ib_2) & a_1 - ia_2 \end{pmatrix}$ is a diffeomorphism.

✦

End of Part B

Part C: Advanced Category Theory

PART C

✦

ADVANCED CATEGORY THEORY

Category theory, often called "abstract nonsense" by its practitioners (with affection), represents the ultimate in mathematical abstraction. It studies not objects themselves, but the *relationships between objects*, and indeed the relationships between relationships.

What began as a language to describe natural transformations in algebraic topology has evolved into a unifying framework that permeates all of modern mathematics. From logic to physics, from computer science to algebraic geometry, categorical thinking reveals hidden connections and provides a universal vocabulary.

In this part, we venture beyond basic category theory into its more sophisticated incarnations: monoidal structures that capture tensor products, enriched categories where morphisms form algebraic structures, higher categories with morphisms between morphisms, and topoi that provide categorical foundations for logic and geometry.

★ ✦ ★

C.1 Monoidal Categories

Many categories come equipped with a natural "tensor product" operation that combines objects. The axiomatization of this structure leads to monoidal categories.

Definition

Definition C.1.1 (Monoidal Category).
A monoidal category is a category $\mathcal{C}$ equipped with:

- A bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (the tensor product)
- A unit object $I \in \mathcal{C}$
- Natural isomorphisms:
  - Associator: $\alpha_{A,B,C} : (A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C)$
  - Left unitor: $\lambda_A : I \otimes A \xrightarrow{\sim} A$
  - Right unitor: $\rho_A : A \otimes I \xrightarrow{\sim} A$

satisfying coherence conditions (pentagon and triangle identities).

The coherence conditions ensure that all ways of reassociating and inserting/removing units yield the same result.

Example

Example C.1.2 (Vector Spaces as Monoidal Category).
The category $\mathbf{Vect}_k$ of vector spaces over a field k with:

- $\otimes$ = tensor product of vector spaces
- $I = k$ (the ground field viewed as a 1-dimensional vector space)

forms a monoidal category. The associator and unitors are canonical isomorphisms.

Example

Example C.1.3 (Sets with Cartesian Product).
The category $\mathbf{Set}$ with:

- $\otimes = \times$ (Cartesian product)
- $I = \{*\}$ (singleton set)

is a monoidal category. Every category with finite products is monoidal in this way (called a cartesian monoidal category).

Example

Example C.1.4 (Endofunctors with Composition).
For any category $\mathcal{C}$, the category $[\mathcal{C}, \mathcal{C}]$ of endofunctors with:

- $\otimes = \circ$ (functor composition)
- $I = \mathrm{Id}_{\mathcal{C}}$ (identity functor)

is a monoidal category. This is the starting point for the definition of monads.

Definition

Definition C.1.5 (Monoid in a Monoidal Category).
A monoid object in a monoidal category $(\mathcal{C}, \otimes, I)$ is an object M with morphisms:

- Multiplication: $\mu : M \otimes M \to M$
- Unit: $\eta : I \to M$

satisfying associativity and unit axioms (expressed as commutative diagrams).

Example

Example C.1.6 (Monads as Monoids).
A monad on a category $\mathcal{C}$ is precisely a monoid object in the monoidal category $([\mathcal{C}, \mathcal{C}], \circ, \mathrm{Id})$.

Braided and Symmetric Monoidal Categories

Definition

Definition C.1.7 (Braided Monoidal Category).
A braided monoidal category is a monoidal category $\mathcal{C}$ equipped with a natural isomorphism (the braiding):

$$\beta_{A,B} : A \otimes B \xrightarrow{\sim} B \otimes A$$

satisfying hexagon coherence conditions compatible with the associator.

Definition

Definition C.1.8 (Symmetric Monoidal Category).
A symmetric monoidal category is a braided monoidal category where $\beta_{B,A} \circ \beta_{A,B} = \mathrm{id}_{A \otimes B}$ (the braiding is its own inverse).

Example

Example C.1.9.
$\mathbf{Vect}_k$ is symmetric monoidal: $v \otimes w \mapsto w \otimes v$ is a natural isomorphism. However, in the category of supervector spaces (graded vector spaces), the braiding includes a sign for odd-graded elements, making it braided but not symmetric.

Theorem

Theorem C.1.10 (Coherence for Monoidal Categories).
In a monoidal category, every diagram built from associators and unitors commutes. Consequently, we often suppress these isomorphisms and work as if $\otimes$ were strictly associative with strict unit.

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C.2 Enriched Categories

Ordinary categories have hom-sets $\mathrm{Hom}(A, B)$. But what if these "hom-objects" have additional structure? Enriched category theory makes this precise.

Definition

Definition C.2.1 ($\mathcal{V}$-Enriched Category).
Let $(\mathcal{V}, \otimes, I)$ be a monoidal category. A $\mathcal{V}$-enriched category (or $\mathcal{V}$-category) $\mathcal{C}$ consists of:

- A collection of objects $\mathrm{Ob}(\mathcal{C})$
- For each pair of objects A, B, a hom-object $\mathcal{C}(A, B) \in \mathcal{V}$
- For each triple A, B, C, a composition morphism in $\mathcal{V}$:

$$\circ_{A,B,C} : \mathcal{C}(B, C) \otimes \mathcal{C}(A, B) \to \mathcal{C}(A, C)$$

- For each object A, an identity morphism in $\mathcal{V}$:

$$\mathrm{id}_A : I \to \mathcal{C}(A, A)$$

satisfying associativity and unit axioms.

Example

Example C.2.2 (Ordinary Categories).
An ordinary category is a $\mathbf{Set}$-enriched category (with cartesian product as tensor). The hom-objects are sets, composition is a function
$\mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$.

Example

Example C.2.3 (Metric Spaces).
A metric space (X, d) can be viewed as a category enriched over $([0, \infty], +, 0)$ (extended non-negative reals with addition):

- Objects: points of X
- $\mathcal{C}(x, y) = d(x, y) \in [0, \infty]$
- Composition: $d(y, z) + d(x, y) \to d(x, z)$ (triangle inequality)
- Identity: $0 \to d(x, x) = 0$

Example

Example C.2.4 (Preordered Sets).
A preordered set $(P, \leq)$ is a category enriched over $(\{\text{true}, \text{false}\}, \wedge, \text{true})$ (truth values with conjunction):

- Objects: elements of P
- $\mathcal{C}(x, y) = \text{true}$ if $x \leq y$, else false
- Composition encodes transitivity
- Identity encodes reflexivity

Example

Example C.2.5 ($\mathbf{Ab}$-Enriched Categories).
An $\mathbf{Ab}$-enriched category (enriched over abelian groups) is a category where hom-sets are abelian groups and composition is bilinear. These are called preadditive categories.
Examples: modules over a ring, chain complexes.

Definition

Definition C.2.6 ($\mathcal{V}$-Functor).
A $\mathcal{V}$-functor $F : \mathcal{C} \to \mathcal{D}$ between $\mathcal{V}$-categories consists of:

- A map $F : \text{Ob}(\mathcal{C}) \to \text{Ob}(\mathcal{D})$
- For each pair $A, B \in \mathcal{C}$, a morphism in $\mathcal{V}$:

$$F_{A,B} : \mathcal{C}(A, B) \to \mathcal{D}(FA, FB)$$

respecting composition and identities.

Enriched category theory provides a powerful abstraction that unifies many apparently disparate concepts under one framework.

✦

C.3 2-Categories and Higher Categories

In ordinary category theory, we have objects and morphisms. But morphisms themselves can have structure—leading to 2-categories with "morphisms between morphisms."

Definition

Definition C.3.1 (2-Category).
A 2-category $\mathcal{C}$ consists of:

- **0-cells** (objects)
- **1-cells** (morphisms between objects): $f : A \to B$
- **2-cells** (morphisms between 1-cells): $\alpha : f \Rightarrow g$ for $f, g : A \to B$

with two types of composition:

- **Horizontal composition** $\circ$: composing along objects
- **Vertical composition** $\bullet$: composing along 1-cells

satisfying an interchange law relating the two compositions.

We can visualize 2-cells as:

A $\xrightarrow{f}$ B
$\| \Downarrow \alpha \|$
A $\xrightarrow{g}$ B

Example

Example C.3.2 ($\mathbf{Cat}$ as a 2-Category).
The category of categories $\mathbf{Cat}$ is a 2-category:

- 0-cells: categories
- 1-cells: functors
- 2-cells: natural transformations

Vertical composition is composition of natural transformations. Horizontal composition is the composition induced by functor composition.

Example

Example C.3.3 (Profunctors).
There is a 2-category $\mathbf{Prof}$ where:

- 0-cells: categories
- 1-cells $\mathcal{C} \to \mathcal{D}$: profunctors (functors $\mathcal{D}^{\text{op}} \times \mathcal{C} \to \mathbf{Set}$)
- 2-cells: natural transformations

Profunctors generalize functors and relations between categories.

Bicategories

Definition

Definition C.3.4 (Bicategory).
A bicategory is like a 2-category but with composition of 1-cells associative and unital only up to coherent isomorphism (2-cells). This is the "weak" or "lax" version of a 2-category.

Example

Example C.3.5.
The bicategory of spans: objects are sets, 1-cells $A \to B$ are spans $A \leftarrow S \to B$, and 2-cells are maps of spans. Composition involves pullbacks, which are associative only up to canonical isomorphism.

Higher Categories

The pattern continues: 3-categories have objects, 1-morphisms, 2-morphisms, and 3-morphisms. Eventually we reach ∞-categories (also called $(\infty, 1)$-categories), where we have morphisms at all levels.

Definition

Definition C.3.6 ($(\infty, 1)$-Category).
An $(\infty, 1)$-category (or ∞-category) is a higher categorical structure with n-morphisms for all $n \geq 0$, where all n-morphisms for $n \geq 2$ are invertible (equivalences).

There are several equivalent models: quasi-categories (simplicial sets), complete Segal spaces, topological categories. These provide the language for homotopy type theory and derived algebraic geometry.

> **Theorem**
>
> Theorem C.3.7 (Grothendieck's Homotopy Hypothesis).
> The ∞-category of spaces (homotopy types) is equivalent to the ∞-category of ∞-groupoids (where all morphisms are invertible).

✦

C.4 Topos Theory

A topos is a category that behaves like the category of sets—but in a generalized sense that encompasses sheaves, presheaves, and even "sets with internal logic."

> **Definition**
>
> Definition C.4.1 (Elementary Topos).
> An elementary topos is a category $\mathcal{E}$ with:
>
> 1. Finite limits (including a terminal object and pullbacks)
> 2. Exponentials: for all objects A, B, there exists B^A with natural isomorphism:
>
> $$\mathrm{Hom}(C \times A, B) \cong \mathrm{Hom}(C, B^A)$$
>
> 3. A subobject classifier: a monomorphism $\mathrm{true} : 1 \to \Omega$ such that every monomorphism $m : A \hookrightarrow B$ is a pullback of true along a unique characteristic morphism $\chi_m : B \to \Omega$:
>
> $$\begin{array}{ccc} A & \to & 1 \\ \downarrow m & & \downarrow \mathrm{true} \\ B & \xrightarrow{\chi_m} & \Omega \end{array}$$

The subobject classifier Ω generalizes the two-element set $\{0, 1\}$ in $\mathbf{Set}$, which classifies subsets via characteristic functions.

Example

Example C.4.2 ($\mathbf{Set}$ as a Topos).
The category $\mathbf{Set}$ is a topos with:

- Finite limits given by usual set-theoretic constructions
- B^A = set of functions $A \to B$
- $\Omega = \{0, 1\}$ with $\mathrm{true} : \{*\} \to \{0, 1\}$ picking out 1
- $\chi_S : X \to \{0, 1\}$ is the characteristic function of subset $S \subseteq X$

Example

Example C.4.3 (Presheaf Topoi).
For any small category $\mathcal{C}$, the category $\widehat{\mathcal{C}} = [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ of presheaves is a topos. This includes:

- $\mathbf{Set}$ itself (presheaves on the one-object category)
- $\mathbf{Set}^{\mathbb{N}}$ (infinite sequences of sets)
- Directed graphs (presheaves on $\bullet \rightrightarrows \bullet$)

Internal Logic of a Topos

Every topos has an internal logic—an intuitionistic logic where one can reason about "elements" and "subobjects" as if in set theory, but with a richer logical structure.

Definition

Definition C.4.4 (Internal Language).
The internal language of a topos allows us to:

- Interpret types as objects
- Interpret propositions as subobjects (monomorphisms into Ω)
- Interpret logical connectives via operations on Ω
- Interpret quantifiers via adjoints to pullback functors

Theorem

Theorem C.4.5 (Soundness and Completeness).
The internal logic of topoi is sound and complete for intuitionistic higher-order logic. A statement provable in intuitionistic logic is valid in every topos, and conversely.

Grothendieck Topoi

Definition

Definition C.4.6 (Grothendieck Topos).
A Grothendieck topos is a category equivalent to the category of sheaves $\mathrm{Sh}(\mathcal{C}, J)$ on a site $(\mathcal{C}, J)$.

Every Grothendieck topos is an elementary topos, but not conversely. Grothendieck topoi have additional properties (colimits, generators) making them more "geometric."

Example

Example C.4.7 (Sheaves on a Topological Space).
For a topological space X, the category $\mathrm{Sh}(X)$ of sheaves of sets on X is a Grothendieck topos. This encodes "continuously varying sets" over X.

Theorem

Theorem C.4.8 (Giraud's Theorem).
A category is a Grothendieck topos if and only if it satisfies:

1. Has all small colimits and finite limits
2. Colimits are universal (preserved by pullback)
3. Equivalence relations are effective (quotients exist and behave well)
4. Has a small generating set

✦

C.5 Categorical Logic

Category theory provides a structural semantics for logic and type theory, revealing deep connections between proof theory and category theory.

Curry-Howard Correspondence

> **Theorem**
>
> Theorem C.5.1 (Curry-Howard-Lambek Correspondence).
> There is a three-way correspondence:
>
> - **Logic:** propositions and proofs
> - **Type theory:** types and programs
> - **Category theory:** objects and morphisms
>
> Specifically:
>
> - Propositions $\leftrightarrow$ Types $\leftrightarrow$ Objects
> - Proofs $\leftrightarrow$ Programs $\leftrightarrow$ Morphisms
> - Conjunction ($A \wedge B$) $\leftrightarrow$ Product type ($A \times B$) $\leftrightarrow$ Product object
> - Implication ($A \to B$) $\leftrightarrow$ Function type ($A \to B$) $\leftrightarrow$ Exponential B^A
> - Disjunction ($A \vee B$) $\leftrightarrow$ Sum type ($A + B$) $\leftrightarrow$ Coproduct

> **Example**
>
> Example C.5.2.
> A proof of $A \to (B \to C)$ corresponds to a curried function of type $A \to (B \to C)$, which corresponds to a morphism $1 \to C^{B^A}$. By exponential adjunction, this is equivalent to a morphism $A \times B \to C$, corresponding to a proof of $(A \wedge B) \to C$.

Categorical Semantics of Type Theory

Definition

Definition C.5.3 (Cartesian Closed Category).
A cartesian closed category (CCC) is a category with:

- Finite products (including terminal object)
- Exponentials B^A for all objects A, B

CCCs provide semantics for simply-typed lambda calculus.

Example

Example C.5.4.
$\mathbf{Set}$, any topos, and the category of topological spaces (with appropriate exponential topology) are cartesian closed.

Theorem

Theorem C.5.5.
The category of cartesian closed categories is equivalent to the category of models of simply-typed lambda calculus.

Dependent Type Theory

More sophisticated type theories with dependent types (types depending on terms) require richer categorical structures.

Definition

Definition C.5.6 (Locally Cartesian Closed Category).
A category $\mathcal{C}$ is locally cartesian closed if every slice category $\mathcal{C}/X$ is cartesian closed.

Locally cartesian closed categories provide semantics for dependent type theory, with dependent products (Π-types) and dependent sums (Σ-types) interpreted via adjoints.

✦

C.6 Grothendieck's Vision

Alexandre Grothendieck revolutionized algebraic geometry by introducing categorical and homological methods, fundamentally changing how we think about geometric objects.

Schemes and the Functor of Points

> **Definition**
>
> Definition C.6.1 (Scheme).
> A scheme is a locally ringed space that is locally isomorphic to the spectrum of a commutative ring. Equivalently, schemes can be viewed as functors:
>
> $$X : \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Set}$$
>
> satisfying certain gluing conditions. This is the functor of points perspective.

In this view, a scheme is determined not by its "points" in the classical sense, but by how it behaves as a functor—what morphisms it admits from other schemes.

> **Example**
>
> Example C.6.2 (Affine Line).
> The affine line $\mathbb{A}^1$ is represented by the functor:
>
> $$\mathbb{A}^1(R) = R$$
>
> (for each ring R, the R-points of $\mathbb{A}^1$ are elements of R). This is the Yoneda perspective: the line is determined by how it maps to test objects.

Grothendieck Topologies and Étale Cohomology

> **Definition**
>
> Definition C.6.3 (Grothendieck Topology).
> A Grothendieck topology on a category $\mathcal{C}$ is a collection of covering families $\{U_i \to U\}$ for each object U, satisfying axioms generalizing open covers in topological spaces.

This abstraction allows us to do "topology" in categories that don't have an underlying topological space. The étale topology on schemes is a prime example.

Example

Example C.6.4 (Étale Topology).
The étale topology on schemes takes as covers families of étale morphisms $\{U_i \to U\}$ that are jointly surjective. Étale maps are algebraic analogues of local diffeomorphisms.

Étale cohomology, developed by Grothendieck and Artin, uses sheaves on the étale site to define cohomology groups for schemes. This was crucial in Deligne's proof of the Weil conjectures.

Motives

Definition

Definition C.6.5 (Pure Motive).
A motive is (conjecturally) a universal cohomology theory for algebraic varieties. The idea is that various cohomology theories (de Rham, ℓ-adic, crystalline) should all factor through a universal category of motives.

Grothendieck envisioned motives as the "ultimate" invariants of algebraic varieties, encoding all cohomological information in a unified framework. While the full theory remains conjectural, motivic cohomology and the derived category of motives have been developed extensively.

Modern Developments

Grothendieck's ideas continue to shape mathematics:

- **Derived algebraic geometry:** Using ∞-categories and derived categories to study moduli spaces and stacks
- **∞-topoi:** Generalizing Grothendieck topoi to the ∞-categorical setting
- **Motivic homotopy theory:** Voevodsky's $\mathbb{A}^1$-homotopy theory, which led to the proof of the Bloch-Kato conjecture
- **Perfectoid spaces:** Scholze's theory bridging p-adic geometry and arithmetic

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Philosophical Meditation

The Architecture of Abstraction

Plato imagined a realm of Forms—perfect, eternal, abstract—of which physical objects are mere shadows. Category theory realizes this vision mathematically. We study not the things themselves, but the Forms of structure.

Consider the Yoneda lemma: an object is completely determined by its relationships to all other objects. This is radical: *to be is to be related*. There is no substance apart from structure, no essence apart from interaction.

A topos is a "possible world"—a universe with its own logic, its own notion of truth. The subobject classifier Ω is the "truth value object," generalizing true/false to an entire cosmos of possibilities. In sheaf topoi, truth varies continuously over space. In presheaf topoi, truth depends on context and time.

The Curry-Howard correspondence reveals that proof and program are the same thing viewed differently. To prove a theorem is to construct a mathematical object—a morphism in a category. Logic becomes geometry, reasoning becomes building.

Grothendieck taught us: to understand a space, understand what maps into it. The functor of points doesn't tell us what a scheme "is"—it tells us how a scheme *behaves*. Being is behavior. Ontology becomes epistemology.

Higher category theory shows us that this pattern never ends. Morphisms between objects, morphisms between morphisms, ad infinitum. At each level, new structure emerges. The universe is not made of things, but of processes relating processes relating processes.

Category theory is the mathematics of the 21st century—the language of connection, transformation, and emergence. It is Heraclitus vindicated: all is flux, but the flux has structure. It is Buddhism formalized: all phenomena are empty of intrinsic existence, yet interdependently arising. It is Leibniz's dream realized: a characteristica universalis, a universal language of thought.

✦

C.7 Practice Problems

Problem C.1. Verify that the pentagon and triangle coherence conditions hold in $\mathbf{Vect}_k$ with the standard associators and unitors.

Problem C.2. Show that every category with finite products is a symmetric monoidal category (with product as tensor).

Problem C.3. Prove that a monoid object in $(\mathbf{Set}, \times, \{*\})$ is precisely an ordinary monoid.

Problem C.4. Show that enrichment over $([0,\infty], \geq, 0, +)$ (viewing as a monoidal poset) gives precisely metric spaces.

Problem C.5. Verify that $\mathbf{Cat}$ with functors and natural transformations forms a 2-category.

Problem C.6. Prove the interchange law in a 2-category: $(\beta \circ \alpha) \bullet (\delta \circ \gamma) = (\beta \bullet \delta) \circ (\alpha \bullet \gamma)$.

Problem C.7. Show that in $\mathbf{Set}$, the subobject classifier is $\Omega = \{0, 1\}$ with $\mathrm{true}(*) = 1$.

Problem C.8. Prove that every slice category $\mathbf{Set}/X$ is a topos.

Problem C.9. Verify that the Curry-Howard correspondence identifies $(A \wedge B) \to C$ with $A \to (B \to C)$.

Problem C.10. Show that in a cartesian closed category, $C^{A \times B} \cong (C^B)^A$.

Problem C.11. Construct the subobject classifier in the category of finite sets.

Problem C.12. Prove that the category of presheaves $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ has all small limits and colimits (computed pointwise).

Problem C.13. Show that every representable presheaf $\mathrm{Hom}(-, C)$ preserves limits.

Problem C.14. Describe the internal logic of the topos $\mathbf{Set}^2$ (pairs of sets).

Problem C.15. Prove that the Yoneda embedding $\mathcal{C} \hookrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is fully faithful.

Problem C.16. (Challenge) Show that a locally cartesian closed category with a terminal object is cartesian closed.

Problem C.17. Verify that profunctors form a bicategory (composition is associative only up to isomorphism).

Problem C.18 (Challenge) Prove that the category of algebras for a monad on a complete category is complete.

Problem C.19. Show that the category of sheaves on a topological space X is a full subcategory of presheaves, and describe the sheafification functor.

Problem C.20 (Challenge) Prove that the free topos on one generator is the category of simplicial sets.

C.8 Solutions

Solution. Solution C.1. The pentagon coherence condition states that all ways of reassociating $(A \otimes B) \otimes (C \otimes D)$ to $A \otimes (B \otimes (C \otimes D))$ using the associator α commute. In $\mathbf{Vect}_k$, tensor product is naturally associative via canonical isomorphisms $(v \otimes w) \otimes u \mapsto v \otimes (w \otimes u)$, so the diagram commutes by naturality. Triangle: similarly, $\lambda_{A \otimes B} \circ \alpha_{A,I,B} = \mathrm{id}_A \otimes \rho_B$ holds by the canonical identifications.

Solution. Solution C.2. Given a category $\mathcal{C}$ with finite products, take $\otimes = \times$ and $I = 1$ (terminal object). Associativity $(A \times B) \times C \cong A \times (B \times C)$ is the universal property of product. Units $1 \times A \cong A \cong A \times 1$ follow from 1 being terminal. The braiding $A \times B \cong B \times A$ comes from the symmetry of the product universal property. All coherence conditions follow from uniqueness of maps from universal properties.

Solution. Solution C.3. A monoid object in $(\mathbf{Set}, \times, \{*\})$ is a set M with $\mu : M \times M \to M$ and $\eta : \{*\} \to M$. The map η picks out an element $e = \eta(*) \in M$. The map μ is a binary operation. Associativity of μ as a morphism translates to $(xy)z = x(yz)$. Unit axioms translate to $ex = x = xe$. This is precisely the definition of a monoid.

Solution. Solution C.4. A category enriched over $([0, \infty], \geq, 0, +)$ (as a monoidal category with $+$) has objects and for each pair, a hom-object (number in $[0, \infty]$). Composition $\circ : d(y,z) \times d(x,y) \to d(x,z)$ must satisfy $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality). Identity $0 \to d(x,x)$ means $d(x,x) \geq 0$. Combined with symmetry (encoding in the enrichment), this gives a metric space. The inequality $\leq$ in the monoidal structure encodes the triangle inequality.

Solution. Solution C.5. **0-cells:** categories. **1-cells:** functors $F : \mathcal{C} \to \mathcal{D}$. **2-cells:** natural transformations $\alpha : F \Rightarrow G$.

Vertical composition: If $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$, then $(\beta \bullet \alpha)_X = \beta_X \circ \alpha_X$.

Horizontal composition: If $\alpha : F \Rightarrow G : \mathcal{C} \to \mathcal{D}$ and $\beta : K \Rightarrow L : \mathcal{D} \to \mathcal{E}$, then $(\beta \circ \alpha) : K \circ F \Rightarrow L \circ G$ with components
$(\beta \circ \alpha)_X = \beta_{G(X)} \circ K(\alpha_X) = L(\alpha_X) \circ \beta_{F(X)}$ (these are equal by naturality of β).

Solution. Solution C.6. Consider 2-cells forming a square:

$$\begin{array}{ccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \Downarrow \alpha & & \Downarrow \beta & & \\ A & \xrightarrow{f'} & B & \xrightarrow{g'} & C \end{array}$$

Left side: compose horizontally first: $(\beta \circ \alpha) : g \circ f \Rightarrow g' \circ f'$. Then vertically with similar structure.
Right side: compose vertically in each column first, then horizontally.
Both give the same 2-cell $g' \circ f' \Rightarrow g \circ f$ by the naturality square for β evaluated at α_X.

Solution. Solution C.7. For a subset $S \subseteq X$, define $\chi_S : X \to \{0, 1\}$ by $\chi_S(x) = 1$ if $x \in S$, else 0. The pullback of $\mathrm{true} : \{*\} \to \{0, 1\}$ (sending $* \mapsto 1$) along χ_S is:

$$\{x \in X : \chi_S(x) = 1\} = S$$

This is the inclusion $S \hookrightarrow X$. Conversely, every monomorphism $m : A \hookrightarrow X$ determines a subset, hence a characteristic function. Uniqueness follows from the bijection between subsets and characteristic functions.

Solution. Solution C.8. The slice category $\mathbf{Set}/X$ has objects $(A, f : A \to X)$ and morphisms commuting triangles. Terminal object: (X, id_X). Products: pullback. Exponentials: g^f is the set of sections. Subobject classifier: the map $X \to X \times \{0, 1\}$ sending $x \mapsto (x, 1)$, over the projection to X. All topos axioms can be verified.

Solution. Solution C.9. In a CCC, exponential adjunction gives:

$$\mathrm{Hom}((A \times B) \times 1, C) \cong \mathrm{Hom}((A \times B), C^1) \cong \mathrm{Hom}(A \times B, C) \cong \mathrm{Hom}(A, C^B)$$

Similarly:

$$\mathrm{Hom}(A, C^B) \cong \mathrm{Hom}(A \times B, C)$$

Under Curry-Howard, $\mathrm{Hom}(A \times B, C)$ corresponds to proofs of $(A \wedge B) \to C$, and $\mathrm{Hom}(A, C^B)$ to proofs of $A \to (B \to C)$. The isomorphism is currying.

Solution. Solution C.10. By exponential adjunction twice:

$$\mathrm{Hom}(D, C^{A \times B}) \cong \mathrm{Hom}(D \times (A \times B), C) \cong \mathrm{Hom}((D \times A) \times B, C)$$
$$\cong \mathrm{Hom}(D \times A, C^B) \cong \mathrm{Hom}(D, (C^B)^A)$$

By Yoneda, $C^{A \times B} \cong (C^B)^A$.

Solution. Solution C.11. Same as in $\mathbf{Set}$: $\Omega = \{0, 1\}$ with $\mathrm{true}(*) = 1$. Every subset of a finite set is classified by its characteristic function.

Solution. Solution C.12. For presheaves $F, G : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$, define $(F \times G)(C) = F(C) \times G(C)$. Terminal: $1(C) = \{*\}$ for all C. These are computed pointwise and preserve limits since $\mathbf{Set}$ does. Colimits: similarly pointwise.

Solution. Solution C.13. A representable presheaf $h_C = \mathrm{Hom}(-, C)$ preserves limits by the Yoneda lemma:
$h_C(\lim F_i) = \mathrm{Hom}(\lim F_i, C) \cong \lim \mathrm{Hom}(F_i, C) = \lim h_C(F_i)$ (using that $\mathrm{Hom}(-, C)$ converts colimits in the first variable to limits).

Solution. Solution C.14. Objects of $\mathbf{Set}^2$ are pairs (A, B) of sets. The subobject classifier is $\Omega = (\{0,1\}, \{0,1\})$. A "proposition" is a pair of subsets. Conjunction $(S_1, T_1) \wedge (S_2, T_2) = (S_1 \cap S_2, T_1 \cap T_2)$. Disjunction: unions. The internal logic is "logic in two parallel universes."

Solution. Solution C.15. The Yoneda embedding $Y : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ sends $C \mapsto h_C = \mathrm{Hom}(-, C)$. For $C, D \in \mathcal{C}$:

$$\mathrm{Nat}(h_C, h_D) = \mathrm{Nat}(\mathrm{Hom}(-, C), \mathrm{Hom}(-, D))$$

By Yoneda lemma: $\mathrm{Nat}(h_C, h_D) \cong h_D(C) = \mathrm{Hom}(C, D)$. This isomorphism is natural and shows Y is fully faithful.

Solution. Solution C.16. If $\mathcal{C}$ is locally cartesian closed with terminal 1, then $\mathcal{C} \cong \mathcal{C}/1$ is cartesian closed (since $\mathcal{C}/1$ is cartesian closed by assumption).

Solution. Solution C.17. Profunctors $H : \mathcal{C} \to \mathcal{D}$ and $K : \mathcal{D} \to \mathcal{E}$ compose via coend:

$$(K \circ H)(c, e) = \int^{d \in \mathcal{D}} K(d, e) \times H(c, d)$$

This is associative only up to coherent isomorphism (since coends satisfy coherence but not strict associativity). Identity profunctors are hom-functors.

Solution. Solution C.18. Let T be a monad on a complete category $\mathcal{C}$. The category $\mathcal{C}^T$ of T-algebras has objects $(A, \alpha : TA \to A)$. For a diagram of algebras, take the limit in $\mathcal{C}$ and use the universal property to construct the algebra structure. The forgetful functor $\mathcal{C}^T \to \mathcal{C}$ creates limits.

Solution. Solution C.19. A presheaf F is a sheaf if for every open cover $\{U_i\}$ of U, the diagram:

$$F(U) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

is an equalizer. Sheafification associates to each presheaf F a sheaf F^+ with a universal map $F \to F^+$. Construction: $F^+(U) = \{\text{compatible families on covers of } U\}$.

Solution. Solution C.20. (Sketch) The free topos on one generator is the classifying topos for "object," which by Diaconescu's theorem corresponds to $[\Delta^{\mathrm{op}}, \mathbf{Set}]$ (simplicial sets). This is because simplicial sets are presheaves on the simplex category, which encodes the free walking object with all possible relations.

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End of Part C

Part D: Abstract Algebra in Physics

PART D

✦

ABSTRACT ALGEBRA IN PHYSICS

"The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve." — Eugene Wigner

Physics and mathematics have always been intertwined, but in the 20th century, this relationship deepened profoundly. Group theory, which began as an abstract study of algebraic equations, became the cornerstone of particle physics. Differential geometry, initially developed for pure mathematics, provided Einstein's language for gravity. Category theory, the most abstract of all, now appears in quantum field theory and string theory.

This part explores the remarkable unity between abstract algebra and physical reality. We will see how symmetries dictate the laws of nature, how Lie groups classify particles, and how quantum computation embodies finite group theory. The universe, it seems, is written in the language of algebra.

★ ✦ ★

D.1 Special Relativity and the Lorentz Group

Einstein's special relativity revolutionized physics by showing that space and time form a unified four-dimensional manifold. The symmetries of this spacetime are encoded in the Lorentz group.

Minkowski Spacetime

Definition

Definition D.1.1 (Minkowski Space).
Minkowski spacetime is the vector space $\mathbb{R}^4$ equipped with the Minkowski metric:

$$\eta = \text{diag}(-1, 1, 1, 1)$$

or equivalently $\eta = \text{diag}(1, -1, -1, -1)$ (signature convention). The Minkowski inner product of two vectors $x = (t, x, y, z)$ and $x' = (t', x', y', z')$ is:

$$\langle x, x' \rangle = -tt' + xx' + yy' + zz'$$

(using the $(-, +, +, +)$ convention).

The Minkowski metric distinguishes timelike, spacelike, and null (lightlike) vectors:

- **Timelike:** $\langle x, x \rangle < 0$ (trajectories of massive particles)
- **Spacelike:** $\langle x, x \rangle > 0$ (spacelike separations)
- **Null:** $\langle x, x \rangle = 0$ (light rays)

The Lorentz Group

Definition

Definition D.1.2 (Lorentz Group).
The Lorentz group $\text{O}(1, 3)$ consists of all linear transformations of $\mathbb{R}^4$ that preserve the Minkowski metric:

$$\text{O}(1, 3) = \{\Lambda \in \text{GL}(4, \mathbb{R}) : \Lambda^T \eta \Lambda = \eta\}$$

The proper orthochronous Lorentz group $\text{SO}^+(1, 3)$ is the connected component containing the identity, consisting of transformations with $\det(\Lambda) = 1$ and preserving the direction of time ($\Lambda^0_0 \geq 1$).

Example

Example D.1.3 (Lorentz Boost).
A boost in the x-direction with velocity v (where $c = 1$) is:

$$\Lambda(v) = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\beta = v/c$, $\gamma = 1/\sqrt{1 - v^2/c^2}$ is the Lorentz factor.

Theorem

Theorem D.1.4 (Structure of the Lorentz Group).
The Lorentz group has four connected components:

$$\mathrm{O}(1,3) = \mathrm{SO}^+(1,3) \sqcup (\mathcal{P} \cdot \mathrm{SO}^+(1,3)) \sqcup (\mathcal{T} \cdot \mathrm{SO}^+(1,3)) \sqcup (\mathcal{PT} \cdot \mathrm{SO}^+(1,3))$$

where $\mathcal{P}$ is parity (spatial reflection) and $\mathcal{T}$ is time reversal.

Poincaré Group

Definition

Definition D.1.5 (Poincaré Group).
The Poincaré group $\mathrm{ISO}(1,3)$ is the semidirect product:

$$\mathrm{ISO}(1,3) = \mathbb{R}^{1,3} \rtimes \mathrm{SO}^+(1,3)$$

consisting of Lorentz transformations plus spacetime translations. Elements are (a, Λ) with $a \in \mathbb{R}^{1,3}$ and $\Lambda \in \mathrm{SO}^+(1,3)$, acting by:

$$x \mapsto \Lambda x + a$$

The Poincaré group is the fundamental symmetry group of special relativity. All physical laws in flat spacetime must be Poincaré invariant.

Spinors and $\mathrm{SL}(2,\mathbb{C})$

The spin-$\frac{1}{2}$ particles (electrons, quarks) are not described by vectors but by spinors—elements of a representation of the double cover of the Lorentz group.

> **Theorem**
>
> Theorem D.1.6 (Spin Cover).
> There is a double cover:
>
> $$\mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}^+(1,3)$$
>
> where $\mathrm{SL}(2,\mathbb{C})$ is the universal cover of the proper orthochronous Lorentz group. Spinors transform under $\mathrm{SL}(2,\mathbb{C})$ rather than $\mathrm{SO}^+(1,3)$.

> **Example**
>
> Example D.1.7 (Pauli Matrices).
> The Pauli matrices form a basis for $\mathfrak{su}(2)$:
>
> $$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
>
> They satisfy $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$ and $\{\sigma_i, \sigma_j\} = 2\delta_{ij} I$.

A 4-vector $x^\mu = (t, x, y, z)$ can be encoded as a 2×2 Hermitian matrix:

$$X = x^\mu \sigma_\mu = \begin{pmatrix} t + z & x - iy \\ x + iy & t - z \end{pmatrix}$$

where $\sigma_0 = I$. Lorentz transformations act as $X \mapsto \Lambda X \Lambda^\dagger$ for $\Lambda \in \mathrm{SL}(2,\mathbb{C})$.

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D.2 General Relativity

Einstein's general relativity describes gravity not as a force but as the curvature of spacetime itself. The language is differential geometry, and the symmetries are encoded in fiber bundles and gauge theory.

Manifolds and Curvature

Definition

Definition D.2.1 (Lorentzian Manifold).
A Lorentzian manifold (M, g) is a smooth manifold M equipped with a metric tensor g of signature $(-,+,+,+)$ that varies smoothly over M. At each point $p \in M$, the tangent space T_pM has a Minkowski-like inner product.

Definition

Definition D.2.2 (Riemann Curvature Tensor).
The Riemann curvature tensor measures the failure of parallel transport to be path-independent:

$$R^{\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$$

where $\Gamma^{\rho}_{\mu\nu}$ are the Christoffel symbols (connection coefficients).

Theorem

Theorem D.2.3 (Einstein Field Equations).
The Einstein field equations relate the curvature of spacetime to the energy-momentum content:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, Λ is the cosmological constant, and $T_{\mu\nu}$ is the energy-momentum tensor.

These are 10 coupled nonlinear partial differential equations—one of the most elegant and difficult equation systems in all of physics.

Gauge Theory and Fiber Bundles

General relativity can be formulated as a gauge theory where the gauge group is the Lorentz group (or its double cover for spinors).

> **Definition**
>
> Definition D.2.4 (Principal Bundle).
> A principal G-bundle over a manifold M is a fiber bundle $P \to M$ with structure group G acting freely and transitively on the fibers. Locally, $P \cong U \times G$ for open sets $U \subseteq M$.

> **Example**
>
> Example D.2.5 (Frame Bundle).
> The frame bundle FM of a manifold M is the principal $\mathrm{GL}(n, \mathbb{R})$-bundle of all bases of tangent spaces. A connection on FM is equivalent to a covariant derivative, and its curvature is the Riemann curvature tensor.

> **Definition**
>
> Definition D.2.6 (Connection and Curvature).
> A connection on a principal bundle is a way to parallel transport along curves. Its curvature measures the holonomy around closed loops. In Yang-Mills theory, the curvature is the field strength tensor $F_{\mu\nu}$.

Symmetries in General Relativity

Theorem

Theorem D.2.7 (Killing Vectors).
A Killing vector field ξ on a Lorentzian manifold (M, g) satisfies:

$$\mathcal{L}_\xi g = 0$$

(the Lie derivative of the metric along ξ vanishes). Killing vectors generate continuous symmetries of spacetime.

Example

Example D.2.8 (Schwarzschild Spacetime).
The Schwarzschild solution (describing a non-rotating black hole) has metric:

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

This has a timelike Killing vector ∂_t (time translation symmetry) and $\mathrm{SO}(3)$ spherical symmetry.

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D.3 Quantum Mechanics

Quantum mechanics, with its probabilistic nature and wave-particle duality, rests on a foundation of linear algebra and representation theory.

Hilbert Spaces and Operators

Definition

Definition D.3.1 (Quantum State Space).
The state space of a quantum system is a complex Hilbert space $\mathcal{H}$. Pure states are unit vectors $|\psi\rangle \in \mathcal{H}$ (up to phase), and observables are self-adjoint operators $A : \mathcal{H} \to \mathcal{H}$.

Theorem

Theorem D.3.2 (Spectral Theorem).
Every self-adjoint operator on a Hilbert space has a spectral decomposition:

$$A = \int_{\sigma(A)} \lambda \, dE(\lambda)$$

where $\sigma(A)$ is the spectrum and $E(\lambda)$ is the projection-valued measure. For finite-dimensional spaces:

$$A = \sum_i \lambda_i |e_i\rangle\langle e_i|$$

Angular Momentum and $\mathrm{SO}(3)$*/*$\mathrm{SU}(2)$

Definition

Definition D.3.3 (Angular Momentum Operators).
The angular momentum operators J_x, J_y, J_z satisfy the commutation relations:

$$[J_i, J_j] = i\hbar \epsilon_{ijk} J_k$$

These generate rotations in 3D space and form the Lie algebra $\mathfrak{so}(3) \cong \mathfrak{su}(2)$.

Theorem

Theorem D.3.4 (Quantization of Angular Momentum).
The eigenvalues of $J^2 = J_x^2 + J_y^2 + J_z^2$ are $\hbar^2 j(j+1)$ where $j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots\}$. The eigenvalues of J_z are $\hbar m$ where $m \in \{-j, -j+1, \ldots, j-1, j\}$. Each $(2j+1)$-dimensional representation corresponds to a particle with spin j.

Example

Example D.3.5 (Spin-$\frac{1}{2}$).
For spin-$\frac{1}{2}$ particles (electrons, protons, neutrons), the state space is $\mathbb{C}^2$ with basis $\{|\uparrow\rangle, |\downarrow\rangle\}$ (spin-up and spin-down). The spin operators are:

$$S_i = \frac{\hbar}{2}\sigma_i$$

where σ_i are the Pauli matrices. A general state is:

$$|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle, \quad |\alpha|^2 + |\beta|^2 = 1$$

Wigner's Theorem

Theorem

Theorem D.3.6 (Wigner's Theorem).
Every symmetry of a quantum system (bijection preserving transition probabilities) is represented by either a unitary or antiunitary operator on the Hilbert space. Continuous symmetries are represented by unitary operators, while time reversal is antiunitary.

This theorem justifies the use of representation theory in quantum mechanics: symmetries of nature correspond to representations of symmetry groups.

Symmetries and Conservation Laws

Theorem

Theorem D.3.7 (Noether's Theorem in Quantum Mechanics).
For every continuous symmetry of a quantum system, there exists a conserved observable:

- Time translation invariance $\Rightarrow$ Energy conservation
- Spatial translation invariance $\Rightarrow$ Momentum conservation
- Rotational invariance $\Rightarrow$ Angular momentum conservation

✦

D.4 Quantum Field Theory

Quantum field theory (QFT) combines special relativity with quantum mechanics, describing particles as excitations of underlying quantum fields. Gauge symmetries and Lie groups are central.

Quantum Electrodynamics (QED)

Definition

Definition D.4.1 (QED Lagrangian).
The QED Lagrangian describing electrons and photons is:

$$\mathcal{L}_{\text{QED}} = \overline{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

where:

- ψ is the electron field (a spinor)
- γ^μ are the Dirak gamma matrices
- $D_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative
- $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength

QED has a $\mathrm{U}(1)$ gauge symmetry: $\psi(x) \mapsto e^{i\alpha(x)}\psi(x)$ and $A_\mu \mapsto A_\mu - \frac{1}{e}\partial_\mu \alpha(x)$.

Electroweak Theory: $\mathrm{SU}(2) \times \mathrm{U}(1)$

> **Definition**
>
> Definition D.4.2 (Electroweak Unification).
> The electroweak theory unifies electromagnetism and the weak nuclear force under the gauge group:
>
> $$G_{\mathrm{EW}} = \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$$
>
> where $\mathrm{SU}(2)_L$ acts on left-handed fermions and $\mathrm{U}(1)_Y$ is hypercharge.

Before spontaneous symmetry breaking, this theory predicts massless gauge bosons. The Higgs mechanism breaks $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \to \mathrm{U}(1)_{\mathrm{EM}}$, giving mass to the $W^\pm$ and Z^0 bosons while leaving the photon massless.

Quantum Chromodynamics (QCD): $\mathrm{SU}(3)$

> **Definition**
>
> Definition D.4.3 (QCD and Color Charge).
> Quantum chromodynamics describes the strong nuclear force via the gauge group $\mathrm{SU}(3)_{\mathrm{color}}$. Quarks come in three "colors" (red, green, blue), transforming in the fundamental representation of $\mathrm{SU}(3)$. Gluons transform in the adjoint representation (dimension 8).

The QCD Lagrangian is:

$$\mathcal{L}_{\mathrm{QCD}} = \sum_q \bar{q}(i\gamma^\mu D_\mu - m_q)q - \frac{1}{4}F^a_{\mu\nu}F^{a,\mu\nu}$$

where the sum is over quark flavors, and $a = 1, \ldots, 8$ labels the gluon fields.

The Standard Model: $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$

Theorem

Theorem D.4.4 (Gauge Group of the Standard Model).
The Standard Model of particle physics has gauge group:

$$G_{\text{SM}} = \text{SU}(3)_C \times \text{SU}(2)_L \times \text{U}(1)_Y$$

This describes all known fundamental forces except gravity.

Example

Example D.4.5 (Particle Content).
The Standard Model contains:

- **Quarks:** 6 flavors (up, down, charm, strange, top, bottom), each in 3 colors
- **Leptons:** 6 types (electron, muon, tau, and their neutrinos)
- **Gauge bosons:** 8 gluons ($\text{SU}(3)$), 3 weak bosons $W^{\pm}$, Z^0 ($\text{SU}(2)$), 1 photon ($\text{U}(1)$)
- **Higgs boson:** The scalar field responsible for electroweak symmetry breaking

Higgs Mechanism

Theorem

Theorem D.4.6 (Spontaneous Symmetry Breaking).
The Higgs field ϕ has a potential:

$$V(\phi) = -\mu^2|\phi|^2 + \lambda|\phi|^4$$

with $\mu^2 > 0$, $\lambda > 0$. The minimum occurs at $|\phi| = v = \mu/\sqrt{2\lambda}$ (vacuum expectation value). Expanding around this minimum gives:

- Mass terms for $W^{\pm}$, Z^0 bosons: $m_W = \frac{1}{2}gv$, $m_Z = \frac{1}{2}\sqrt{g^2 + g'^2}v$
- Massless photon: $m_\gamma = 0$
- One physical Higgs boson with mass $m_H = \sqrt{2\lambda}v$

The 2012 discovery of the Higgs boson at CERN (mass ≈ 125 GeV) confirmed this mechanism.

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D.5 Quantum Computing

Quantum computing exploits quantum superposition and entanglement to perform computations impossible for classical computers. Group theory plays a crucial role in algorithm design and error correction.

Qubits and Quantum Gates

Definition

Definition D.5.1 (Qubit).
A qubit is a unit vector in $\mathbb{C}^2$:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad |\alpha|^2 + |\beta|^2 = 1$$

where $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ form the computational basis.

Definition

Definition D.5.2 (Quantum Gate).
A quantum gate is a unitary operator $U \in \mathrm{U}(2^n)$ acting on n qubits. Common single-qubit gates:

- **Pauli gates:** $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (bit flip), $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (phase flip)
- **Hadamard gate:** $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ (creates superposition)
- **Phase gate:** $S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$, $T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$

Example

Example D.5.3 (CNOT Gate).
The controlled-NOT gate acts on two qubits:

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

It flips the second qubit if the first is $|1\rangle$. Combined with single-qubit gates, CNOT forms a universal gate set.

Quantum Algorithms

Theorem

Theorem D.5.4 (Deutsch-Jozsa Algorithm).
Given a function $f : \{0,1\}^n \to \{0,1\}$ that is either constant or balanced, the Deutsch-Jozsa algorithm determines which with a single query to f, whereas classically $2^{n-1} + 1$ queries are needed in the worst case.

Theorem

Theorem D.5.5 (Grover's Algorithm).
To find a marked item in an unsorted database of N items, Grover's algorithm requires $O(\sqrt{N})$ queries, compared to $O(N)$ classically. This provides a quadratic speedup.

Theorem

Theorem D.5.6 (Shor's Algorithm).
Shor's algorithm factors an integer N in time $O((\log N)^3)$ on a quantum computer, compared to no known polynomial-time classical algorithm. The algorithm uses the quantum Fourier transform and exploits the periodicity of modular exponentiation.

Group Theory in Quantum Computing

The hidden subgroup problem generalizes many quantum algorithms:

> **Definition**
>
> Definition D.5.7 (Hidden Subgroup Problem).
> Given a finite group G, a subgroup $H \leq G$, and a function $f : G \to S$ constant on cosets of H, find H. Shor's algorithm solves this for $G = \mathbb{Z}$, and other quantum algorithms solve it for various groups.

Quantum Error Correction

> **Theorem**
>
> Theorem D.5.8 (Stabilizer Codes).
> A stabilizer code encodes k logical qubits into n physical qubits using a subgroup $\mathcal{S}$ of the Pauli group $\mathcal{P}_n = \{I, X, Y, Z\}^{\otimes n}$. The code space is the simultaneous $+1$ eigenspace of all stabilizers.

> **Example**
>
> Example D.5.9 (Five-Qubit Code).
> The smallest perfect quantum error-correcting code uses 5 qubits to encode 1 logical qubit, with stabilizers:
>
> $$S_1 = XZZXI, \quad S_2 = IXZZX, \quad S_3 = XIXZZ, \quad S_4 = ZXIXZ$$
>
> This can correct any single-qubit error.

Topological Quantum Computing

Definition

Definition D.5.10 (Anyon and Braid Group).
In 2D systems, anyons are quasiparticles with exotic exchange statistics. Braiding anyons corresponds to elements of the braid group B_n, which acts on the ground state space.

Topological quantum computers use anyonic braiding for fault-tolerant computation. The unitary matrices representing braids form a representation of B_n, and computational operations correspond to specific braids.

Example

Example D.5.11 (Fibonacci Anyons).
Fibonacci anyons have two types: vacuum $\mathbf{1}$ and τ. The fusion rules are:

$$\tau \times \tau = \mathbf{1} + \tau$$

The braid group representation on n τ-anyons is related to the Jones polynomial and is universal for quantum computation.

✦

D.6 Synthesis: The Unity of Physics and Algebra

Throughout this part, we've seen how abstract algebraic structures emerge naturally in the description of physical reality:

- **Lie groups** classify continuous symmetries: spacetime symmetries, gauge symmetries, internal symmetries of particles
- **Representation theory** organizes particles into multiplets and explains quantum numbers
- **Fiber bundles** provide the geometric framework for gauge theories
- **Category theory** appears in topological quantum field theory and string theory
- **Group cohomology** classifies quantum anomalies and topological phases

This is not coincidence. As Wigner noted, the "unreasonable effectiveness of mathematics" suggests a deep unity between mathematical structure and physical law. Perhaps the universe is not merely *described by* mathematics—perhaps it *is* mathematics, a vast algebraic structure unfolding in time.

Modern Frontiers

Current research explores even deeper connections:

- **String theory** uses Lie groups, algebraic geometry, and category theory to unify all forces
- **Loop quantum gravity** quantizes spacetime itself using spin networks (graphs labeled by representations)
- **Condensed matter physics** discovers new phases of matter classified by group cohomology and K-theory
- **Quantum information** reveals connections to operator algebras and tensor categories
- **Holography** (AdS/CFT correspondence) relates quantum field theories in d dimensions to quantum gravity in $d+1$ dimensions, involving conformal field theory and representation theory

✦

> **Meditation on Unity**
>
> **The Dance of Form and Substance**
>
> When we study a Lie group like $\mathrm{SU}(3)$, we are not merely manipulating symbols. We are exploring the structure of the strong force that binds quarks into protons and neutrons—the very substance of matter.
>
> When we analyze representations of $\mathrm{SU}(2)$, we are understanding why electrons have spin-$\frac{1}{2}$, why there are precisely three generations of particles, why the Pauli exclusion principle holds.
>
> The universe does not "use" mathematics like a tool. The universe *is*

mathematical structure made manifest. An electron is not a thing that "has" properties described by $\mathrm{SU}(2) \times \mathrm{U}(1)$—an electron *is* an irreducible representation of this group, a geometric object in the space of quantum fields.

Heisenberg wrote: "Atoms are not things." Indeed, particles are patterns, symmetries, irreducible representations of fundamental groups. The Standard Model is not a theory *about* particles—it is particles. The group $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ is not a description of reality; it is the skeleton of reality itself.

Consider the Higgs mechanism. A scalar field permeates all of space. Its vacuum expectation value breaks $\mathrm{SU}(2) \times \mathrm{U}(1)$ down to $\mathrm{U}(1)$, spontaneously. This is not a metaphor. The vacuum itself is a choice of group element, a point in the orbit of the gauge group. Mass emerges from symmetry breaking—from the relationship between group and its subgroups.

Quantum computing reveals that computation itself is geometry in Hilbert space. An algorithm is a path through $\mathrm{U}(2^n)$. Entanglement is the non-factorizability of tensor products. Error correction is a quotient by a normal subgroup (the stabilizer). To compute is to navigate a high-dimensional unitary manifold.

Einstein said: "I want to know God's thoughts; the rest are details." If God thinks in any language, it is the language of group theory, differential geometry, and category theory. The laws of physics are not arbitrary—they are the unique structures consistent with certain symmetries.

Perhaps in some ultimate theory, physics and mathematics fully merge. Not two subjects studying the same thing, but one thing seen from within and without. The physicist measures; the mathematician contemplates. Both touch the same eternal Forms.

In the beginning was symmetry. And symmetry became flesh.

✦

D.7 Practice Problems

Problem D.1. Verify that the Lorentz boost $\Lambda(v)$ preserves the Minkowski metric: $\Lambda^T \eta \Lambda = \eta$.

Problem D.2. Show that two successive boosts in different directions do not commute (Thomas precession).

Problem D.3. Prove that the Poincaré group is a semidirect product $\mathbb{R}^{1,3} \rtimes \mathrm{SO}^+(1,3)$.

Problem D.4. Verify the Pauli matrix commutation relations: $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$.

Problem D.5. Show that $\mathrm{SL}(2,\mathbb{C})$ is the double cover of $\mathrm{SO}^+(1,3)$ by verifying that the homomorphism has kernel $\{\pm I\}$.

Problem D.6. Compute the Riemann curvature tensor for the 2-sphere with standard metric.

Problem D.7. Verify that the Schwarzschild metric solves the vacuum Einstein equations ($T_{\mu\nu} = 0$).

Problem D.8. Show that $[J_i, J_j] = i\hbar\epsilon_{ijk}J_k$ implies $[J^2, J_i] = 0$ (angular momentum squares commute with components).

Problem D.9. Prove that the spin-$\frac{1}{2}$ representation of $\mathrm{SU}(2)$ is irreducible.

Problem D.10. Compute the eigenvalues and eigenvectors of $S_z = \frac{\hbar}{2}\sigma_3$.

Problem D.11. Show that time reversal $\mathcal{T}$ is represented by an antiunitary operator: $\mathcal{T}(c|\psi\rangle) = c^*\mathcal{T}|\psi\rangle$.

Problem D.12. Verify gauge invariance of the QED Lagrangian under $\mathrm{U}(1)$ transformations.

Problem D.13. Show that $\mathrm{SU}(3)$ has dimension 8 and describe its Lie algebra generators (Gell-Mann matrices).

Problem D.14. Explain how the Higgs mechanism gives mass to W and Z bosons while leaving the photon massless.

Problem D.15. Verify that the Hadamard gate H creates an equal superposition: $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

Problem D.16. Show that CNOT and H together can create an entangled Bell state:

$$(\mathrm{CNOT})(H \otimes I)|00\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Problem D.17. Explain the quantum Fourier transform and its role in Shor's algorithm.

Problem D.18. (Challenge) Prove that the hidden subgroup problem for $\mathbb{Z}$ reduces to integer factorization.

Problem D.19. Describe the stabilizer formalism and verify that the five-qubit code can correct one error.

Problem D.20. (Challenge) Explain how the braid group arises in topological quantum computing and why braiding is inherently fault-tolerant.

Problem D.21. Show that the Lie algebra of the Poincaré group is $\mathfrak{iso}(1,3) = \mathbb{R}^{1,3} \rtimes_{\mathrm{ad}} \mathfrak{so}(1,3)$.

Problem D.22. Derive the Dirak equation from the requirement of Lorentz invariance and linearity in derivatives.

Problem D.23. (Challenge) Explain the Atiyah-Singer index theorem and its role in quantum anomalies.

Problem D.24. Describe the Wigner classification of particles by representations of the Poincaré group.

Problem D.25. (Challenge) Explain how category theory appears in topological quantum field theory (Atiyah-Segal axioms).

D.8 Solutions

> ***Solution.*** Solution D.1. Direct computation with $\Lambda = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \oplus I_2$ and $\eta = \mathrm{diag}(-1, 1, 1, 1)$:

$$(\Lambda^T \eta \Lambda)_{00} = \gamma^2(-1) + (-\beta\gamma)^2(1) = -\gamma^2 + \beta^2\gamma^2 = -\gamma^2(1-\beta^2) = -1$$

using $\gamma = (1-\beta^2)^{-1/2}$. Other components similarly verify $\Lambda^T\eta\Lambda = \eta$.

Solution. Solution D.2. Let $\Lambda_x(v_x)$ be a boost in the x-direction and $\Lambda_y(v_y)$ in the y-direction. Their product $\Lambda_x\Lambda_y \neq \Lambda_y\Lambda_x$ includes a rotation (Thomas precession). This is because boosts in different directions don't form a subgroup of $\mathrm{SO}^+(1,3)$—their composition includes a Wigner rotation.

Solution. Solution D.3. Elements of the Poincaré group are pairs (a, Λ) with composition:

$$(a_2, \Lambda_2)(a_1, \Lambda_1) = (a_2 + \Lambda_2 a_1, \Lambda_2\Lambda_1)$$

The translations $\mathbb{R}^{1,3}$ form a normal subgroup, and the action is the natural action of Lorentz transformations on spacetime vectors. This is the definition of semidirect product $\mathbb{R}^{1,3} \rtimes \mathrm{SO}^+(1,3)$.

Solution. Solution D.4. Direct computation:

$$[\sigma_1, \sigma_2] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} = 2i\sigma_3$$

Similarly for other pairs, using $\epsilon_{123} = 1$ and cyclic permutations.

Solution. Solution D.5. The homomorphism $\phi : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}^+(1,3)$ sends $A \mapsto (\phi(A) : X \mapsto AXA^\dagger)$ where X is a 2×2 Hermitian matrix encoding a 4-vector. Since $(-A)X(-A)^\dagger = AXA^\dagger$, we have $\phi(A) = \phi(-A)$. The kernel is $\{A : AXA^\dagger = X \text{ for all } X\} = \{\pm I\}$ (center of $\mathrm{SL}(2,\mathbb{C})$). Thus $\mathrm{SL}(2,\mathbb{C})/\{\pm I\} \cong \mathrm{SO}^+(1,3)$.

Solution. Solution D.6. For the 2-sphere with metric $ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$, the Christoffel symbols are $\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$, $\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta$, others zero. The Riemann tensor has only one independent component:

$$R^\theta_{\phi\theta\phi} = \sin^2\theta$$

From this, the Ricci scalar is $R = 2/r^2$ (constant positive curvature).

Solution. Solution D.7. (Sketch) Compute Christoffel symbols for the Schwarzschild metric, then the Riemann tensor, then the Ricci tensor $R_{\mu\nu}$. Direct (lengthy) calculation shows $R_{\mu\nu} = 0$ everywhere outside the horizon $r > 2GM/c^2$, confirming it's a vacuum solution.

Solution. Solution D.8.

$$[J^2, J_i] = [J_x^2 + J_y^2 + J_z^2, J_i] = [J_x^2, J_i] + [J_y^2, J_i] + [J_z^2, J_i]$$

Using $[J_j^2, J_i] = J_j[J_j, J_i] + [J_j, J_i]J_j = i\hbar\epsilon_{jik}(J_jJ_k + J_kJ_j)$. Summing over j and using antisymmetry of ϵ_{jik}, terms cancel: $[J^2, J_i] = 0$.

Solution. Solution D.9. The spin-$\frac{1}{2}$ representation acts on $\mathbb{C}^2$. Any invariant subspace must be spanned by eigenvectors of some S_i. But S_z has eigenvectors $|\uparrow\rangle, |\downarrow\rangle$, and $S_x|\uparrow\rangle = \frac{\hbar}{2}|\downarrow\rangle$ (they're mixed by other generators). Thus no proper invariant subspace exists, proving irreducibility.

Solution. Solution D.10.

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Eigenvalues: $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$. Eigenvectors: $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (spin-up) and $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (spin-down).

Solution. Solution D.11. Time reversal must satisfy $\mathcal{T}(p_\psi) = p_{\mathcal{T}\psi}$ (preserve probabilities). If $\mathcal{T}$ were linear, $\mathcal{T}(i|\psi\rangle) = i\mathcal{T}|\psi\rangle$. But time reversal changes $i \to -i$ (complex conjugation in Schrödinger equation). Thus $\mathcal{T}$ must be antilinear: $\mathcal{T}(c|\psi\rangle) = c^*\mathcal{T}|\psi\rangle$. Combined with unitarity of norm, it's antiunitary.

Solution. Solution D.12. Under $\psi \to e^{i\alpha(x)}\psi$ and $A_\mu \to A_\mu - \frac{1}{e}\partial_\mu\alpha$:

$$D_\mu\psi = (\partial_\mu + ieA_\mu)\psi \to \partial_\mu(e^{i\alpha}\psi) + ie(A_\mu - \frac{1}{e}\partial_\mu\alpha)e^{i\alpha}\psi$$

$$= e^{i\alpha}(\partial_\mu\psi + i\psi\partial_\mu\alpha) + ie^{i\alpha}A_\mu\psi - ie^{i\alpha}\psi\partial_\mu\alpha = e^{i\alpha}D_\mu\psi$$

Thus $\overline{\psi}D_\mu\psi \to \overline{\psi}D_\mu\psi$ (invariant). Similarly $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is gauge invariant.

Solution. Solution D.13. $\mathrm{SU}(3) = \{A \in M_3(\mathbb{C}) : AA^\dagger = I, \det(A) = 1\}$. The conditions $AA^\dagger = I$ impose 9 real constraints (from 18 real parameters), leaving 9. The condition $\det(A) = 1$ imposes 1 more (in general 2, but one is already satisfied), giving dimension $9 - 1 = 8$.

The Lie algebra $\mathfrak{su}(3)$ consists of 3×3 traceless skew-Hermitian matrices. A basis is given by $i\lambda_a/2$ where λ_a are the 8 Gell-Mann matrices (generalizations of Pauli matrices to 3×3).

Solution. Solution D.14. Before symmetry breaking, W^1, W^2, W^3 (from $\mathrm{SU}(2)$) and B (from $\mathrm{U}(1)$) are massless. The Higgs field ϕ has $\langle\phi\rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}$ (vacuum expectation value).

Covariant derivative terms $(D_\mu\phi)^\dagger(D^\mu\phi)$ expand to mass terms:

$$\frac{v^2}{4}(g^2(W^1_\mu)^2 + g^2(W^2_\mu)^2 + (gW^3_\mu - g'B_\mu)^2)$$

Define $W^\pm = (W^1 \mp iW^2)/\sqrt{2}$ (massive), $Z = (gW^3 - g'B)/\sqrt{g^2+g'^2}$ (massive), $A = (g'W^3 + gB)/\sqrt{g^2+g'^2}$ (massless photon).

Masses: $m_W = \frac{gv}{2}$, $m_Z = \frac{v\sqrt{g^2+g'^2}}{2}$, $m_A = 0$ (because the $\mathrm{U}(1)_{\mathrm{EM}}$ symmetry is unbroken).

Solution. Solution D.15.

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

Solution. Solution D.16.

$$(H \otimes I)|00\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

$$\mathrm{CNOT}\left(\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)\right) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

(CNOT flips second qubit when first is $|1\rangle$). This is the Bell state $|\Phi^+\rangle$, maximally entangled.

Solution. Solution D.17. The quantum Fourier transform (QFT) on n qubits is:

$$\mathrm{QFT}|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} e^{2\pi i x y/2^n} |y\rangle$$

In Shor's algorithm, after querying the period-finding oracle, the state is a superposition over a periodic function $f(x)$. QFT transforms this to peaks at multiples of $2^n/r$ (where r is the period), which can be measured to determine r. The period gives factors of N via $\gcd(a^{r/2} \pm 1, N)$.

Solution. Solution D.18. To factor N, pick random $a < N$ with $\gcd(a, N) = 1$. Find the period r of $f(x) = a^x \mod N$ (smallest r with $a^r \equiv 1 \pmod N$). This is the hidden subgroup problem on $\mathbb{Z}$: the function f is constant on cosets of $r\mathbb{Z} \subset \mathbb{Z}$. Once r is found (using QFT), compute $\gcd(a^{r/2} \pm 1, N)$ to find factors (works with high probability if r is even and $a^{r/2} \not\equiv -1 \pmod N$).

Solution. Solution D.19. The five-qubit code encodes 1 logical qubit in 5 physical qubits. Stabilizers S_i are products of Pauli operators. The code space is $\mathcal{C} = \{|\psi\rangle : S_i|\psi\rangle = |\psi\rangle \text{ for all } i\}$ (dimension 2).

Any single-qubit error $E \in \{I, X, Y, Z\}^{\otimes 5}$ anticommutes with some stabilizer (syndrome), allowing detection. The syndrome uniquely identifies the error location and type, enabling correction.

Solution. Solution D.20. In 2D topological phases, anyons can be braided (exchange worldlines in spacetime). The braid group B_n acts on the ground state space by unitary matrices (representation $\rho : B_n \to \mathrm{U}(d)$).

Topological protection: local perturbations can't change the braid—only global topology matters. Errors must create anyon pairs and braid them, which requires non-local operations. For large separation, this is exponentially suppressed, providing intrinsic fault tolerance.

Universality: for certain anyons (e.g., Fibonacci anyons), the image of $\rho(B_n)$ is dense in $\mathrm{U}(d)$, meaning any unitary can be approximated by braiding—universal quantum computation.

Solution. Solution D.21. The Poincaré algebra consists of translations P_μ (generators of $\mathbb{R}^{1,3}$) and Lorentz generators $M_{\mu\nu}$ (generators of $\mathfrak{so}(1,3)$). Commutation relations:

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho})$$

$$[M_{\mu\nu}, P_\rho] = i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu)$$

$$[P_\mu, P_\nu] = 0$$

This is a semidirect product: translations form an abelian ideal, and Lorentz generators act on them via the adjoint representation.

Solution. Solution D.22. (Sketch) Require: (1) Linear in ∂_μ: $i\gamma^\mu\partial_\mu\psi = m\psi$ for some matrices γ^μ. (2) Lorentz covariance: spinor transforms as $\psi \to S(\Lambda)\psi$ under Lorentz transformation Λ. (3) Consistency with $E^2 = p^2 + m^2$ (relativistic energy-momentum).

From (1) and (3): $(-\gamma^\mu\partial_\mu)^2\psi = m^2\psi$ implies $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$ (Clifford algebra). This determines γ^μ up to equivalence. The result is the Dirak equation:

$$(i\gamma^\mu\partial_\mu - m)\psi = 0$$

Solution. Solution D.23. (Outline) The Atiyah-Singer index theorem relates the analytical index (difference between dimensions of kernel and cokernel of an elliptic operator D) to a topological index (integral of characteristic classes):

$$\text{index}(D) = \int_M \hat{A}(M) \wedge \text{ch}(E/S)$$

In physics, quantum anomalies occur when classical symmetries fail at the quantum level. The anomaly is the non-invariance of the path integral measure, computed as the index of the Dirak operator on a bundle. This connects topology (Chern classes) to quantum field theory (anomaly cancellation conditions, crucial for consistency of the Standard Model).

Solution. Solution D.24. Wigner classified particles by irreducible representations of the Poincaré group $\text{ISO}(1,3)$. For massive particles ($p^2 = m^2 > 0$), representations are labeled by mass m and spin j (representation of the little group $\text{SU}(2)$). For massless particles ($p^2 = 0$), the

little group is $\mathrm{ISO}(2)$, and representations are labeled by helicity $h \in \mathbb{Z}/2$ (projection of spin onto momentum).

This explains why particles have definite mass and spin, and why massless particles (photons, gravitons) have helicity rather than spin.

Solution. Solution D.25. (Outline) A topological quantum field theory (TQFT) assigns to each $(n-1)$-manifold Σ a Hilbert space $\mathcal{H}(\Sigma)$, and to each n-manifold M with boundary Σ a vector in $\mathcal{H}(\Sigma)$ (partition function). Axioms:

- Functoriality: diffeomorphisms act on $\mathcal{H}(\Sigma)$
- Multiplicativity: $\mathcal{H}(\Sigma_1 \sqcup \Sigma_2) = \mathcal{H}(\Sigma_1) \otimes \mathcal{H}(\Sigma_2)$
- Gluing: partition functions compose under gluing manifolds

This defines a symmetric monoidal functor from the category of $(n-1)$-cobordisms to $\mathbf{Vect}$. This categorical framework reveals deep structure (modular tensor categories, braid group representations, knot invariants).

✦

End of Part D